NAME:

MATH 835: Evolutionary Partial Differential Equations

Fall'12 | In-class Quiz #2 | October 19

Warm up questions

1. (10 points) Write the solution of the initial value probem

$$u_t + 2u_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \qquad u(x, 0) = \sin x \qquad x \in \mathbb{R}.$$

2. (10 points) Write the Burgers equation $u_t + u\, u_x = 0$ in the form of a conservation law. (i.e., what is the flux function q for this model?)

3. (10 points) Consider the equation

$$u_t + uu_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \qquad u(x, 0) = \exp(x) \qquad x \in \mathbb{R}.$$

Compute all the characteristic curves and show (graphically) that they fill the upper half plane without ever intersecting.

4. (10 points) If the characteristics of a scalar conservation law are the lines

$$x = (1 + \tfrac{1}{3}t)\xi \qquad \xi \in \mathbb{R} \text{ is a parameter,}$$

and the initial condition is $u(x, 0) = x^3$, what is the value of u at the point $(x, t) = (1, 3)$?

The main problem

We are going to examine carefully some of the more or less painful details concerning weak and strong solutions. The origin of the story is a hyperbolic conservation law

$$u_t + q(u)_x = 0 \qquad \text{or equivalently} \qquad u_t + q'(u)u_x = 0. \tag{1}$$

where $q : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^2$ and we assume that there exists a continuously differentiable function r such that

$$r(q'(x)) = x \qquad \forall x. \tag{2}$$

The function r is defined in the interval of all possible wave speeds of the model:

$$W := (\inf_{\xi\in\mathbb{R}} q'(\xi), \sup_{\xi\in\mathbb{R}} q'(\xi)) = (w_{\inf}, w_{\sup})$$

To fill the gaps with rarefaction waves starting at the point $(x_0, 0)$, we propose the function

$$u_{\text{fill}}(x,t) := r\left(\frac{x - x_0}{t}\right),$$

defined on the cone $\{(x,t) : \frac{x-x_0}{t} \in W\}$.

(a) (10 points) Show that u_{fill} is a strong solution of equation (1) wherever it is defined. (This is an easy computation by hand. No need to use complicated arguments.)

(b) (10 points) Take now two constant values $\{u_+, u_-\}$ such that

$$\tfrac{1}{2} = q'(u_-) < q'(u_+) = 1.$$

and consider the function

$$g(x) = \begin{cases} u_-, & \text{if } x < x_0, \\ u_+, & \text{if } x > x_0. \end{cases}$$

Compute all the characteristics generated by this initial condition. Make a plot of the characteristics, with equal scaling in the (x,t) plane (i.e., make the real slopes of the lines visible).

(c) (10 points) **A solution with a rarefaction wave.** Consider the three regions

$$\begin{aligned} O_- &:= \{(x,t) \in \mathbb{R} \times (0,\infty) \,:\, x < x_0 + \tfrac{t}{2}\}, \\ O_m &:= \{(x,t) \in \mathbb{R} \times (0,\infty) \,:\, x_0 + \tfrac{t}{2} < x < x_0 + t\}, \\ O_+ &:= \{(x,t) \in \mathbb{R} \times (0,\infty) \,:\, x > x_0 + t\}, \end{aligned}$$

and the function

$$u(x,t) = \begin{cases} u_-, & \text{in } O_-, \\ u_{\text{fill}}(x,t), & \text{in } O_m, \\ u_+ & \text{in } O_+. \end{cases}$$

Show that u is a solution of (1) in O_-, in O_+ and in O_m. (Note that this has laready been done in O_m.).

(d) (10 points) Show that the function u in (c) is continuous in $\mathbb{R}\times(0,\infty)$. Continuity is clear in each of the three regions. You only need to show it across the two interfaces. For instance, to show continuity in points in the line between O_m and O_+, it suffices (why?) to show that if $(x,t)\in O_m$ and $(x,t)\to(x_0+\widetilde{t},\widetilde{t})$, then $u_{\rm fill}(x,t)\to u_+$.

(e) (10 points + 10 extra points) Take now $v\in\mathcal{D}(\mathbb{R}\times(0,\infty))$. Detail the argument that shows that the function u in (c) satisfies

$$\int_{-\infty}^{\infty}\int_0^{\infty}(uv_t+q(u)v_x)\mathrm{d}x\mathrm{d}t=0. \tag{3}$$

Hint: you can always choose values $A,\varepsilon>0$ such that the support of v is contained in the rectangle $(x_0-A,x_0+A)\times(\varepsilon,A-\varepsilon)$. This will separate the integration domain into three clean cut areas. It'll help you to plot them, give them a name, name the interfaces, plot the normal vectors on the interfaces, etc.

(f) (10 points + 10 extra points) **A solution with a shock wave.** Consider the number

$$w_0:=\frac{q(u_+)-q(u_-)}{u_+-u_-}.$$

Prove that

$$\tfrac{1}{2}<w_0<1.$$

(To help you a little bit with this, from condition (2) you can get to show that q'' never vanishes and therefore q' is monotonic.) Show that the function

$$u_{\rm bad}(x,t):=\begin{cases}u_- & \text{if } x<x_0+w_0t,\\ u_+ & \text{if } x>x_0+w_0t.\end{cases}$$

satisfies (3) as well.

NAME:

MATH 835: Evolutionary Partial Differential Equations

Fall'12 Take home exam #1 Due October 8

Instructions.

- Write the solution to different problems on separate pages. Two two-sided pages per problem should be enough.
- You can work with *one of your colleagues*. In that case, write a single report of your results. Other than that, do not discuss these results with anyone.

---

1. (30 points) We have proved in class that if $v_t - D\Delta v = 0$ and we define

$$u(\mathbf{x},t) := v(\mathbf{x},t)e^{\frac{1}{2D}\mathbf{x}\cdot\mathbf{b}-(\frac{1}{4D}|\mathbf{b}|^2+c)t}$$

(for constants c, D and a constant vector $\mathbf{b}$), then

$$u_t - D\Delta u + \mathbf{b}\cdot\nabla u + cu = 0.$$

   (a) Use this idea to construct a solution of the initial value problem

$$u_t - D\Delta u + \mathbf{b}\cdot\nabla u + cu = 0 \text{ in } \mathbb{R}^n\times(0,\infty) \qquad u(\cdot,0) = g \text{ in } \mathbb{R}^n,$$

   where $g:\mathbb{R}^n\to\mathbb{R}$ is a bounded integrable function. (You do not need to repeat the arguments we used to construct the solution of the Cauchy problem for the heat equation.)

   (b) Use the previously computed solution to figure out what the fundamental solution associated to the above diffusion-convection-reaction problem is.

2. (30 points) Find an explicit formula for the solution of the problem

$$\left[\begin{array}{ll} u_t = u_{xx} & \text{in } (0,\infty)\times(0,\infty), \\ u(\cdot,0) = g & \text{in } [0,\infty), \\ -u_x(0,t) = 0 & t>0, \end{array}\right.$$

   where $g:(0,\infty)\to\mathbb{R}$ is a bounded function.

3. (40 points) Solve Problem 2.16 of the book. (In addition to the hint given in the book, note that although the result is stated in a different way, there's this possible argument: if the function is non-negative in the parabolic boundary and negative at a certain point, it has a (local) negative minimum.)

4. (50 points) What follows is a breakdown of Problem 2.3 in the book (correcting several typos in the statement).

(a) Give a graphic/analytic argument showing that the algebraic equation

$$\mu \tan \mu = 1$$

has a countable number of solutions that diverge to infinity. Let $\{\mu_k : k \geq 1\}$ be these values

(b) Relate the solution of (a) to the eigenvalue problem

$$-U'' = \lambda U \text{ in } (0,1), \qquad U'(0) = U'(1) + U(1) = 0.$$

(c) We are next going to assume that any square integrable function $g : (0,1) \to \mathbb{R}$ can be written as a series

$$g(x) = \sum_{k=1}^{\infty} c_k \cos(\mu_k x).$$

Use an orthogonality argument to give a formula to compute the coefficients c_k. (You can try by hand, but there's a very simple general argument that shows that eigenfunctions corresponding to different eigenvalues are orthogonal. You should try and use it.) While you do this, prove that

$$\frac{1}{2} \leq \int_0^1 \cos^2(\mu_k x) \mathrm{d}x \leq 1 \qquad \forall k.$$

You'll need it later.

(d) Give a formal solution of the problem

$$\left[\begin{array}{ll} u_t = D u_{xx} & \text{in } (0,1) \times (0,\infty), \\ u(\cdot, 0) = g & \text{in } (0,1), \\ -u_x(0,t) = u(1,t) + u_x(1,t) = 0 & t > 0. \end{array}\right.$$

(e) Use the results of (c) and (d) to show that the formal solution you got in (d) is actually a continuous function in $[0,1] \times (0,\infty)$, *just assuming that g is a bounded function.* (You will need to show that the coefficients c_k are bounded as $k \to \infty$.)

NAME:

MATH 835: Evolutionary Partial Differential Equations

Fall'12 | In-class Quiz #1 | September 21

1. (10 points) This first question is just to warm up and to be sure that we all know what we are dealing with. Here are a scalar function of three variables and a vector field in the space:

$$u := e^{x+2y}\cos(z) \qquad \mathbf{p} := (x^2 y,\, \cos(x+z),\, -z^2+y)$$

Compute Δu and $\nabla \cdot \mathbf{p} = \operatorname{div} \mathbf{p}$.

2. (15 points) Consider a function $u : \Omega \times (0,\infty) \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ and a function $f : \Omega \to \mathbb{R}$. We say that $u(\cdot, 0) = f$ weakly when

$$\lim_{t\to 0} \int_\Omega |u(\mathbf{x},t) - f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 0.$$

Show that if $u(\cdot,0) = f$ weakly and $v(\cdot,0) = g$ weakly, then $(u+v)(\cdot,0) = f+g$ weakly.

3. (15 points) Using the weak maximum principle show that if

$$w \in \mathcal{C}^{2,1}(Q_T) \cap \mathcal{C}(\overline{Q}_T) \qquad Q_T := \Omega \times (0,T)$$

satisfies $w_t - D\Delta w = 0$ in Q_T, then

$$\min_{\partial_p Q_T} w \le w(\mathbf{x},t) \le \max_{\partial_p Q_T} w \qquad \forall (\mathbf{x},t) \in \overline{Q}_T,$$

where $\partial_p Q_T$ is the parabolic boundary of Q_T. Are points of the form $(\mathbf{x}, T)$ with $\mathbf{x} \in \partial\Omega$ part of the parabolic boundary or not? (Give a good reason for your answer. You might find it in the result you have been just asked to prove or in the formulation of the weak maximum principle.)

4. (15 points) In the derivation of the heat kernel, we find ourselves in the situation of having to solve the differential equation

$$U''(\xi) + \tfrac{1}{2}\xi U'(\xi) + \tfrac{1}{2}U(\xi) = 0 \qquad \xi \in \mathbb{R}.$$

Detail the argument where we show that this equation has at least an even solution and that this solution satisfies

$$U'(\xi) + \tfrac{1}{2}\xi U(\xi) = 0 \qquad \forall \xi > 0.$$

(This solution is just any multiple of $\exp(-\frac{1}{4}\xi^2)$, but this is not known at this point.)

5. (15 points) Knowing that

$$u(x,t) := \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

is a solution of $u_t - D u_{xx} = 0$, prove that

$$v(x,y,t) := \frac{1}{4\pi D t} e^{-\frac{x^2+y^2}{4Dt}}$$

is a solution of $v_t - D\Delta v = 0$. Do not do a long computation. Use a simple argument instead.

6. (15 points) Consider the $n-$dimensional heat kernel

$$\Gamma(\mathbf{x},t) := \frac{1}{(4\pi D t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4Dt}}$$

and a bounded continuous function $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$. Show that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \Gamma(\mathbf{y}-\mathbf{x},t) f(\mathbf{y},t) \mathrm{d}\mathbf{y} = f(\mathbf{x},0).$$

(Give all necessary details.)

7. (15 points) Let $u \in \mathcal{C}^{2,1}(\overline{\Omega} \times (0,T))$ be a solution of $u_t - D\Delta u = 0$ in $\Omega \times (0,T)$ satisfying a homogeneous boundary condition of Neumann type. Show that the energy

$$E(t) := \frac{1}{2} \int_{\Omega} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}$$

is non-increasing.