Mathematical Analysis tools for the inquiring numerical analyst

by F.–J. Sayas

Version of February 13, 2013

1 Cut-off functions

Prerequesites and notes. Definition of support of a continuous function and of the set $\mathcal{D}(\Omega)$, of $\mathcal{C}^\infty$ compactly supported functions. The material of this section is standard, there's really nothing new to it.

Radially symmetric bumps. Consider the function

$$g(r) := \begin{cases} e^{\frac{1}{r^2-1}}, & |r| < 1, \\ 0, & |r| \ge 1. \end{cases} \tag{1}$$

Then

$$g \in \mathcal{C}^\infty(\mathbb{R}), \qquad \operatorname{supp} g = [-1,1], \qquad g \ge 0.$$

Therefore, the function $\varphi : \mathbb{R}^d \to \mathbb{R}$

$$\varphi(\mathbf{x}) := g(|\mathbf{x}|) \tag{2}$$

satisfies

$$\varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \operatorname{supp} \varphi = \overline{B(\mathbf{0};1)}, \qquad \varphi \ge 0.$$

With translations and dilations we can thus construct non-ngeative functions in $\mathcal{D}(\mathbb{R}^d)$ with support on $\overline{B(\mathbf{x}_0;\rho)}$ for all $\mathbf{x}_0 \in \mathbb{R}^d$ and $\rho > 0$.

Approximations of the Dirac delta. Let

$$\psi \in \mathcal{D}(\mathbb{R}^d), \qquad \psi \ge 0, \qquad \operatorname{supp} \psi = \overline{B(\mathbf{0};1)}, \qquad \int_{\mathbb{R}^d} \psi(\mathbf{x}) \mathrm{d}\mathbf{x} = 1. \tag{3}$$

Such a function can be easily constructed by dividing the function φ in (2) by its norm. We can then define

$$\psi_\varepsilon(\mathbf{x}) := \tfrac{1}{\varepsilon^d} \psi\left(\tfrac{1}{\varepsilon}\mathbf{x}\right) \qquad \varepsilon > 0. \tag{4}$$

Then

$$\psi_\varepsilon \in \mathcal{D}(\mathbb{R}^d), \qquad \psi_\varepsilon \geq 0, \qquad \operatorname{supp}\psi_\varepsilon = \overline{B(\mathbf{0};\varepsilon)}, \qquad \int_{\mathbb{R}^d} \psi_\varepsilon(\mathbf{x})\mathrm{d}\mathbf{x} = 1.$$

Before thinking of distributions, it is simple to see that if $f \in \mathcal{C}(\mathbb{R})$, then

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \psi_\varepsilon(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} = f(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^d.$$

A smooth Heaviside function. Let g be the function defined in (1). Define then

$$h(r) := \frac{1}{\int_{-\infty}^{\infty} g(2t-1)\mathrm{d}t} \int_{-\infty}^{r} g(2t-1)\mathrm{d}t. \tag{5}$$

Then

$$h \in \mathcal{C}^\infty(\mathbb{R}), \qquad 0 \leq h \leq 1, \qquad h' \geq 0, \tag{6a}$$

and

$$h \equiv 0 \quad \text{in } (-\infty, 0], \qquad h \equiv 1 \quad \text{in } [1,\infty). \tag{6b}$$

Radially symmetric cut-off functions. Consider the function h defined in (5) (or, for the same price, any function satisfying (6)). Then, the function

$$\varphi(\mathbf{x}) := h\left(\frac{R_2 - |\mathbf{x}-\mathbf{x}_0|}{R_2 - R_1}\right), \qquad \mathbf{x}_0 \in \mathbb{R}^d, \qquad 0 < R_1 < R_2,$$

satisfies

$$\varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \operatorname{supp}\varphi = \overline{B(\mathbf{x}_0; R_2)} \qquad \varphi \equiv 1 \text{ in } \overline{B(\mathbf{x}_0; R_1)}, \qquad 0 \leq \varphi \leq 1.$$

Cut-off functions on cubes. Consider now two concentric parallelepideds

$$Q := [-m_1, m_1] \times \ldots \times [-m_d, m_d] \qquad Q^{\mathrm{ext}} := [-M_1, M_1] \times \ldots \times [-M_d, M_d],$$

where

$$0 < m_j < M_j \quad \forall j.$$

Then, the function

$$\varphi(\mathbf{x}) := \prod_{j=1}^{d} h\left(\frac{M_j - x_j}{M_j - m_j}\right)$$

satisfies

$$\varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \operatorname{supp}\varphi = \overline{Q^{\mathrm{ext}}}, \qquad \varphi \equiv 1 \text{ in } \overline{Q}, \qquad 0 \leq \varphi \leq 1.$$

Once again, we can change the center of the set Q to get new functions centered at arbitrary points in $\mathbb{R}^d$.

2 Lattice partitions of unity

Prerequesites and notes. Section 1 of this document. Theorem 2 is a simplified version of Theorem 1.4.4. in Hörmander [1].

A smooth tiling of the space. Let us start by considering a basic configuration of two concentric $d-$cubes

$$Q_{\mathbf{0}} := (-\tfrac{1}{2}, \tfrac{1}{2})^d \qquad Q_{\mathbf{0}}^\delta := (-\tfrac{1}{2} - \delta, \tfrac{1}{2} + \delta)^d$$

and a function

$$\varphi_{\mathbf{0}} \in \mathcal{D}(\mathbb{R}^d), \qquad \operatorname{supp} \varphi_{\mathbf{0}} = \overline{Q_{\mathbf{0}}^\delta}, \qquad \varphi_{\mathbf{0}} \equiv 1 \text{ in } \overline{Q}_\delta, \qquad 0 \le \varphi_{\mathbf{0}} \le 1.$$

Consider then the cubes

$$Q_{\mathbf{n}} := \mathbf{n} + Q_{\mathbf{0}} = \prod_{j=1}^d (n_j - \tfrac{1}{2}, n_j + \tfrac{1}{2}), \qquad Q_{\mathbf{n}}^\delta := \mathbf{n} + Q_{\mathbf{0}}^\delta, \qquad \mathbf{n} \in \mathbb{Z}^d,$$

and the functions

$$\varphi_{\mathbf{n}} := \varphi(\cdot - \mathbf{n}).$$

The sum of all these functions

$$s := \sum_{\mathbf{n} \in \mathbb{Z}^d} \varphi_{\mathbf{n}}$$

satisfies

$$s \in \mathcal{C}^\infty(\mathbb{R}^d), \qquad s > 0, \qquad s \text{ is 1-periodic in all variables.}$$

We finally get to our goal functions:

$$\psi_{\mathbf{n}} := s^{-1} \varphi_{\mathbf{n}} \in \mathcal{D}(\mathbb{R}^d), \qquad 0 \le \psi_{\mathbf{n}} \le 1, \qquad \operatorname{supp} \psi_{\mathbf{n}} = \overline{Q_{\mathbf{n}}^\delta}, \qquad \sum_{\mathbf{n} \in \mathbb{Z}^d} \psi_{\mathbf{n}} \equiv 1.$$

By taking δ small enough, we can enforce $\psi_{\mathbf{n}}$ to be identically equal to one in a neighborhood of $\mathbf{n}$.

Theorem 1 (Smooth separation from the boundary)**.** *Let Ω be an open set and $\varepsilon > 0$. Then, there exists*

$$\varphi \in \mathcal{D}(\Omega), \qquad s.t. \qquad \varphi \equiv 1 \textit{ in } \Omega_\varepsilon := \{\mathbf{x} \in \Omega \,:\, \operatorname{dist}(\mathbf{x}, \partial\Omega) > \varepsilon\}.$$

Proof. Let $\ell_\varepsilon := \varepsilon/(2\sqrt{d})$ and consider the cubes

$$Q_{\mathbf{n}} := \ell_\varepsilon \mathbf{n} + (-\tfrac{1}{2}\ell_\varepsilon, \tfrac{1}{2}\ell_\varepsilon)^d \subset Q_{\mathbf{n}}^{\text{ext}} := \ell_\varepsilon \mathbf{n} + (-\tfrac{2}{3}\ell_\varepsilon, \tfrac{2}{3}\ell_\varepsilon)^d, \qquad \mathbf{n} \in \mathbb{Z}^d.$$

A simple change of scale in the previous construction allows us to get

$$\psi_{\mathbf{n}} \in \mathcal{D}(\mathbb{R}^d) \qquad \operatorname{supp} \psi_{\mathbf{n}} \subset Q_{\mathbf{n}}^{\text{ext}}, \qquad \sum_{\mathbf{n} \in \mathbb{Z}^d} \psi_{\mathbf{n}} \equiv 1.$$

(note that we are asking for $\psi_{\mathbf{n}}$ to be supported inside $Q_{\mathbf{n}}^{\text{ext}}$, which can be easily accomplished by choosing δ small enough in the previous construction.) Consider now the finite set of indices

$$\mathcal{I} := \{\mathbf{n} \in \mathbb{Z}^d \,:\, Q_{\mathbf{n}}^{\text{ext}} \cap \Omega_\varepsilon \neq \emptyset\}$$

and the function

$$\varphi := \sum_{\mathbf{n}\in\mathcal{I}} \psi_{\mathbf{n}} \in \mathcal{D}(\mathbb{R}^d).$$

What is left is just a collection of easy observations.

(a) If $\mathbf{x} \in \Omega_\varepsilon$, then $B(\mathbf{x};\varepsilon) \subset \Omega$.

(b) The diameter of $Q_{\mathbf{n}}^{\text{ext}}$ is $\sqrt{d}\,\frac{4}{3}\,\ell_\varepsilon = \frac{2}{3}\varepsilon$. Therefore, if $\mathbf{n} \in \mathcal{I}$, we can take $\mathbf{x} \in Q_{\mathbf{n}}^{\text{ext}} \cap \Omega_\varepsilon$ and note that

$$\mathbf{y} \in Q_{\mathbf{n}}^{\text{ext}} \quad\Longrightarrow\quad |\mathbf{y}-\mathbf{x}| < \tfrac{2}{3}\varepsilon \quad\Longrightarrow\quad \mathbf{y} \in B(\mathbf{x};\varepsilon) \subset \Omega.$$

(c) As a consequence,

$$\operatorname{supp}\varphi \subset \bigcup_{\mathbf{n}\in\mathcal{I}} Q_{\mathbf{n}}^{\text{ext}} \subset \Omega.$$

(d) Finally, if $\mathbf{x} \in \Omega_\varepsilon$, then $\psi_{\mathbf{n}}(\mathbf{x}) = 0$ for all $\mathbf{n} \notin \mathcal{I}$ and thus

$$\varphi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathcal{I}} \psi_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathcal{Z}^d} \psi_{\mathbf{n}}(\mathbf{x}) = 1.$$

This finishes the proof. □

Theorem 2 (Break down of test functions into cubic lattices)**.** *Let Ω be any open set and $\phi \in \mathcal{D}(\Omega)$. Then, there exist cubes Q_j and $\phi_j \in \mathcal{D}(\Omega)$ with*

$$\phi = \sum_{j=1}^{N} \phi_j, \qquad \operatorname{supp}\phi_j \subset Q_j \subset \Omega.$$

Proof. The proof uses the construction of the previous proof. We start by defining $\varepsilon := \frac{1}{2}\text{dist}(\operatorname{supp}\phi, \partial\Omega)$ so that $\operatorname{supp}\phi \subset \Omega_\varepsilon$ (see Theorem 1). Then we have a finite set of indices $\mathcal{I}$, cubes $\{Q_{\mathbf{n}} \,:\, \mathbf{n} \in \mathcal{I}\}$, and smooth functions

$$\psi_{\mathbf{n}} \in \mathcal{D}(\mathbb{R}^d), \qquad \operatorname{supp}\psi_{\mathbf{n}} \subset \Omega, \qquad \sum_{\mathbf{n}\in\mathcal{I}} \psi_{\mathbf{n}} \equiv 1 \text{ in } \Omega_\varepsilon.$$

Therefore

$$\phi = \phi \sum_{\mathbf{n}\in\mathcal{I}} \psi_{\mathbf{n}} = \sum_{\mathbf{n}\in\mathcal{I}} (\phi\,\psi_{\mathbf{n}}),$$

and the result follows by taking $\varphi_{\mathbf{n}} := \phi\,\varphi_{\mathbf{n}}$ and renumbering the set $\mathcal{I}$. □

3 When the gradient of a distribution vanishes

Prerequesites and notes. Sections 1 and 2 of this document. Concept of distribution and differentiation of distributions. The proof of the one-dimensional version of the theorem can be found in any classic textbook. The $d-$dimensional case is more involved. The proof I'm giving below is distilled from Hörmander's first volume [1]. In particular Proposition 4 below follows [1, Theorem 3.1.4'].

Goal. The aim of this section is the proof of the following important result:

> If $\Omega \subset \mathbb{R}^d$ is an open connected set, then
>
> $$T \in \mathcal{D}'(\Omega), \quad \nabla T = 0 \qquad \Longrightarrow \qquad T = c,$$
>
> i.e., if the partial derivatives of a distribution vanish, then there exists a constant c such that
>
> $$\langle T, \varphi \rangle = c \int_\Omega \varphi(\mathbf{x}) \mathrm{d}\mathbf{x} \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

We will proceed as follows: prove it in one dimension, prove it for cubes, use a tiling argument to prove it for all domains.

Proposition 3. *If $T \in \mathcal{D}'(a,b)$ and $T' = 0$, there exists $c \in \mathbb{R}$ such that*

$$\langle T, \varphi \rangle = c \int_a^b \varphi(x) \mathrm{d}x \qquad \forall \varphi \in \mathcal{D}(a,b).$$

Proof. This results from a very well known argument, that can be found in any basic textbook. Take first

$$\varphi_0 \in \mathcal{D}(a,b) \qquad \int_a^b \varphi_0(t) \mathrm{d}t = 1.$$

Given $\varphi \in \mathcal{D}(a,b)$, it then follows that

$$\begin{aligned} \varphi(t) &= \varphi(t) - \left(\int_a^b \varphi(\tau) \mathrm{d}\tau \right) \varphi_0(t) + \left(\int_a^b \varphi(\tau) \mathrm{d}\tau \right) \varphi_0(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \left(\varphi(\tau) - \Big(\int_a^b \varphi(\rho) \mathrm{d}\rho \Big) \varphi_0(\tau) \right) \mathrm{d}\tau + \left(\int_a^b \varphi(\tau) \mathrm{d}\tau \right) \varphi_0(t) \\ &= \psi'(t) + \left(\int_a^b \varphi(\tau) \mathrm{d}\tau \right) \varphi_0(t), \qquad\qquad \psi \in \mathcal{D}(a,b). \end{aligned}$$

Therefore

$$\langle T, \varphi \rangle = \langle T, \psi' \rangle + \int_a^b \varphi(t) \mathrm{d}t \, \langle T, \varphi_0 \rangle = -\langle T', \psi \rangle + \langle T, \varphi_0 \rangle \int_a^b \varphi(t) \mathrm{d}t,$$

and the result follows by taking $c := \langle T, \varphi_0 \rangle$. $\square$

Proposition 4. *Let $Q := (a_1, b_1) \times \ldots \times (a_d, b_d)$. If*

$$\langle \partial_{x_j} T, \phi \rangle = 0 \qquad \forall \phi \in \mathcal{D}(Q), \quad \forall j,$$

then there exists $c \in \mathbb{R}$ such that

$$\langle T, \phi \rangle = c \int_Q \phi(\mathbf{x}) \mathrm{d}\mathbf{x} \qquad \forall \phi \in \mathcal{D}(Q).$$

Proof. The result is proved by induction on d. We have already proved it for $d = 1$. Let $\widetilde{Q} := (a_1, b_1) \times \ldots \times (a_{d-1}, b_{d-1})$.

We start with

$$\varphi_0 \in \mathcal{D}(a_d, b_d) \qquad \int_{a_d}^{b_d} \phi_0(t) \mathrm{d}t = 1$$

and consider $\widetilde{T} : \mathcal{D}(\widetilde{Q}) \to \mathbb{R}$ given by

$$\langle \widetilde{T}, \widetilde{\varphi} \rangle := \langle T, \widetilde{\varphi} \otimes \varphi_0 \rangle,$$

where

$$(\widetilde{\varphi} \otimes \varphi_0)(\mathbf{x}) = \widetilde{\varphi}(\widetilde{\mathbf{x}})\, \varphi_0(x_d) \qquad \mathbf{x} = (\widetilde{\mathbf{x}}, x_d) \in Q = \widetilde{Q} \times (a_d, b_d).$$

It is very easy to prove that $\widetilde{T} \in \mathcal{D}'(\widetilde{Q})$ and that for $0 \le j \le d-1$

$$\begin{aligned} \langle \partial_{x_j} \widetilde{T}, \widetilde{\varphi} \rangle &= -\langle \widetilde{T}, \partial_{x_j} \widetilde{\varphi} \rangle \\ &= -\langle T, \partial_{x_j} \widetilde{\varphi} \otimes \varphi_0 \rangle \\ &= -\langle T, \partial_{x_j} (\widetilde{\varphi} \otimes \varphi_0) \rangle \\ &= \langle \partial_{x_i} T, \widetilde{\varphi} \otimes \varphi_0 \rangle = 0, \qquad \forall \widetilde{\varphi} \in \mathcal{D}(\widetilde{Q}) \end{aligned}$$

and therefore there exists a constant c such that $\widetilde{T} = c$. On the other hand, if we associate

$$\varphi \in \mathcal{D}(Q) \qquad \longmapsto \qquad \widetilde{\varphi} := \int_{a_d}^{b_d} \varphi(\cdot, t) \mathrm{d}t \in \mathcal{D}(\widetilde{Q})$$

and decompose (see the proof of Proposition 3, where $\widetilde{\varphi}$ is just a value)

$$\varphi = \varphi - \widetilde{\varphi} \otimes \varphi_0 + \widetilde{\varphi} \otimes \varphi_0 = \partial_{x_d} \psi + \widetilde{\varphi} \otimes \varphi_0,$$

where

$$\psi(\mathbf{x}) = \psi(\widetilde{\mathbf{x}}, x_d) := \int_{a_d}^{x_d} \Big(\varphi(\widetilde{\mathbf{x}}, t) - \widetilde{\varphi}(\widetilde{\mathbf{x}})\, \varphi_0(t) \Big) \mathrm{d}t, \qquad \psi \in \mathcal{D}(Q),$$

we show that

$$\begin{aligned} \langle T, \varphi \rangle &= \langle T, \partial_{x_d} \psi \rangle + \langle T, \widetilde{\varphi} \otimes \varphi_0 \rangle = -\langle \partial_{x_d} T, \psi \rangle + \langle T, \widetilde{\varphi} \otimes \varphi_0 \rangle = \langle \widetilde{T}, \widetilde{\varphi} \rangle \\ &= c \int_{\widetilde{Q}} \widetilde{\varphi}(\widetilde{\mathbf{x}}) \mathrm{d}\widetilde{\mathbf{x}} = c \int_Q \varphi(\mathbf{x}) \mathrm{d}\mathbf{x} \end{aligned}$$

and the proof is finished. □

Proposition 5. *Let Ω be open and connected and let $T \in \mathcal{D}'(\Omega)$ satisfy $\nabla T = 0$. Then there exists $c \in \mathbb{R}$ such that*

$$\langle T, \varphi \rangle = c \int_\Omega \varphi(\mathbf{x}) \mathrm{d}\mathbf{x} \qquad \forall \varphi \in \mathcal{D}(\Omega), \qquad \operatorname{supp} \varphi \subset Q,$$

where Q is any cube contained in Ω and c does not depend on Q.

Proof. Proposition 4 shows that for all Q there exists c_Q such that

$$\langle T, \varphi \rangle = c_Q \int_\Omega \varphi(\mathbf{x}) \mathrm{d}\mathbf{x} \qquad \forall \varphi \in \mathcal{D}(\Omega), \qquad \operatorname{supp} \varphi \subset Q.$$

If $Q_1 \cap Q_2 \neq \emptyset$, we choose

$$\phi \in \mathcal{D}(Q_1 \cap Q_2) \qquad \int_\Omega \phi(\mathbf{x}) \mathrm{d}\mathbf{x} = 1$$

and then note that

$$c_{Q_1} = c_{Q_1} \int_\Omega \phi(\mathbf{x}) \mathrm{d}\mathbf{x} = \langle T, \phi \rangle = c_{Q_2}.$$

Let now Q_a and Q_b be any two non-intersecting cubes in Ω. We can find a finite sequence of cubes

$$Q_a = Q_1, Q_2, \ldots, Q_N = Q_b, \qquad Q_j \cap Q_{j+1} \neq \emptyset, \qquad Q_j \subset \Omega.$$

(This is done by a connection and compactness argument. We join the center of the cubes with a continuous arc, associate a cube to each point of the arc, and choose a finite subcover of the arc using compactness.) The previous argument shows then that $c_{Q_j} = c_{Q_{j+1}}$ and therefore $c_{Q_a} = c_{Q_b}$. □

Theorem 6. *Let Ω be open and connected and assume that $T \in \mathcal{D}'(\Omega)$ satisfies*

$$\partial_{x_j} T = 0 \qquad 1 \leq j \leq d.$$

Then $T = c$.

Proof. This is an easy consequence of Proposition 5 and the decompositions of Proposition 2. Given $\phi \in \mathcal{D}(\Omega)$ we can write

$$\phi = \sum_{j=1}^N \phi_j, \qquad \phi_j \in \mathcal{D}(Q_j) \qquad Q_j \subset \Omega,$$

where Q_j are cubes. We then use Proposition 5 to show that

$$\langle T, \phi \rangle = \sum_j \langle T, \phi_j \rangle = \sum_j c \int_\Omega \phi_j = c \int_\Omega \phi,$$

which proves the result. □

4 A guided tour of mollification techniques

Prerequesites and notes. The functions of Section 1 and basic notions of differentiation of distributions. Only hints for the quite easy proofs of this section are given.

Proposition 7. *Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and let*

$$(u * \varphi)(\mathbf{x}) := \int_{\mathbb{R}^d} \varphi(\mathbf{x}-\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}.$$

Then:

$$u * \varphi \in \mathcal{C}(\mathbb{R}^d) \qquad \text{and} \qquad \partial_{x_i}(u * \varphi) = u * \partial_{x_i}\varphi.$$

Therefore $u * \varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$. Additionally,

$$\operatorname{supp}\varphi \subset \overline{B(\mathbf{0};\varepsilon)} \quad \text{and} \quad u \equiv 0 \text{ in } B(\mathbf{x};2\varepsilon) \qquad \Longrightarrow \qquad u * \varphi \equiv 0 \text{ in } B(\mathbf{x};\varepsilon).$$

Proof. Assume that $\operatorname{supp}\varphi \subset B(\mathbf{0};R_\varphi)$. Then

$$(u * \varphi)(\mathbf{x}) = \int_{B(\mathbf{x};R_\varphi)} \varphi(\mathbf{x}-\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y} = \int_{B(\mathbf{0};R+R_\varphi)} \varphi(\mathbf{x}-\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y} \qquad \mathbf{x} \in B(\mathbf{0};R).$$

Note that $u \in L^1(B(\mathbf{0};R+R_\varphi))$. With this at hand, it is easy to prove that $u * \varphi$ is continuous at any point of the ball $B(\mathbf{0};R+R_\varphi)$. Differentiability can also be shown to hold under integral sign. The last property is easy to verify. □

Proposition 8. *Let*

$$\varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \varphi \ge 0, \qquad \int_{\mathbb{R}^d} \varphi(\mathbf{x})\mathrm{d}\mathbf{x} = 1.$$

If $u \in L^2(\mathbb{R}^d)$, then $u * \varphi \in L^2(\mathbb{R}^d)$ and

$$\|u * \varphi\|_{\mathbb{R}^d} \le \|u\|_{\mathbb{R}^d}.$$

Proof. The following inequalities are easy to justify:

$$\begin{aligned}\int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \varphi(\mathbf{y})u(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y}\Big|^2 \mathrm{d}\mathbf{x} &\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(\mathbf{z})\mathrm{d}\mathbf{z}\right)\left(\int_{\mathbb{R}^d} \varphi(\mathbf{y})|u(\mathbf{x}-\mathbf{y})|^2\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^d} \varphi(\mathbf{y})\left(\int_{\mathbb{R}^d} |u(\mathbf{x}-\mathbf{y})|^2 \mathrm{d}\mathbf{x}\right)\mathrm{d}\mathbf{y}.\end{aligned}$$

The result then follows readily. □

Proposition 9. *Let*

$$\varphi \in \mathcal{D}(\mathbb{R}^d), \qquad \varphi \ge 0, \qquad \int_{\mathbb{R}^d} \varphi(\mathbf{x})\mathrm{d}\mathbf{x} = 1.$$

If $u \in H^1(\mathbb{R}^d)$, then

$$\partial_{x_i}(u * \varphi) = (\partial_{x_i}u) * \varphi.$$

Therefore $u * \varphi \in H^1(\mathbb{R}^d)$.

Proof. By Proposition 7, in the classical way $\partial_{x_i}(u * \varphi) = u * \partial_{x_i}\varphi$. Fix now $\mathbf{x} \in \mathbb{R}^d$, consider the function $\varphi_{\mathbf{x}} := \varphi(\mathbf{x} - \cdot) \in \mathcal{D}(\mathbb{R}^d)$ and note that

$$(\partial_{x_i}\varphi)(\mathbf{x} - \,\cdot\,) = -\partial_{x_i}\varphi_{\mathbf{x}}.$$

Then

$$\begin{aligned}(u * \partial_{x_i}\varphi)(\mathbf{x}) &= \int_{\mathbb{R}^d} u(\mathbf{y})(\partial_{x_i}\varphi)(\mathbf{x} - \mathbf{y})\mathrm{d}\mathbf{y} = -\int_{\mathbb{R}^d} u(\mathbf{y})\,\partial_{x_i}\varphi_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{y} \\ &= -\langle u, \partial_{x_i}\varphi_{\mathbf{x}}\rangle = \langle \partial_{x_i}u, \varphi_{\mathbf{x}}\rangle = \int_{\mathbb{R}^d} (\partial_{x_i}u)(\mathbf{y})\varphi_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^d} (\partial_{x_i}u)(\mathbf{y})\varphi(\mathbf{x} - \mathbf{y})\mathrm{d}\mathbf{y} = (\partial_{x_i}u * \varphi)(\mathbf{x}).\end{aligned}$$

By Proposition 8, it follows that $\partial_{x_i}(u * \varphi) \in L^2(\mathbb{R}^d)$. □

Lemma 10 (Translations are continuous in L^2)**.** *For all $u \in L^2(\mathbb{R}^d)$*

$$u(\cdot - \mathbf{h}) \stackrel{|\mathbf{h}|\to 0}{\longrightarrow} u \qquad \text{in } L^2(\mathbb{R}^d).$$

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. It is then easy to show that $\varphi(\cdot - \mathbf{h}) \to \varphi$ uniformly and therefore in $L^2(\mathbb{R}^d)$. For general u, we bound

$$\begin{aligned}\|u(\cdot - \mathbf{h}) - u\|_{\mathbb{R}^d} &\le \|u(\cdot - \mathbf{h}) - \varphi(\cdot - \mathbf{h})\| + \|\varphi(\cdot - \mathbf{h}) - \varphi\|_{\mathbb{R}^d} + \|\varphi - u\|_{\mathbb{R}^d} \\ &= 2\|u - \varphi\|_{\mathbb{R}^d} + \|\varphi(\cdot - \mathbf{h}) - \varphi\|_{\mathbb{R}^d}.\end{aligned}$$

We next notice that the variational lemma implies that

$$\mathcal{D}(\mathbb{R}^d)^\perp = \{u \in L^2(\mathbb{R}^d) \,:\, (u, \varphi)_{\mathbb{R}^d} = 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d)\} = \{0\}$$

and therefore $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. The proof then follows readily. □

Proposition 11. *Let*

$$\varphi \in \mathcal{D}(\mathbb{R}^d), \qquad 0 \le \varphi \le 1, \qquad \int_{\mathbb{R}^d} \varphi(\mathbf{x})\mathrm{d}\mathbf{x} = 1$$

and consider the functions

$$\varphi_\varepsilon := \tfrac{1}{\varepsilon^d}\varphi\Big(\tfrac{1}{\varepsilon}\,\cdot\Big)$$

Then

$$u * \varphi_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} u \qquad \text{in } L^2(\mathbb{R}^d).$$

Proof. Consider the function $\omega(\cdot; u) : \mathbb{R}^d \to [0, \infty)$ defined by

$$\omega(\mathbf{h}; u) := \int_{\mathbb{R}^d} |u(\mathbf{x} - \mathbf{h}) - u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}.$$

Note that $\omega(\cdot; u)$ is continuous at zero by Lemma 10. Also

$$\omega(\mathbf{h}; u) = \omega(\mathbf{h} - \widehat{\mathbf{h}}; u(\cdot + \widehat{\mathbf{h}})) \qquad \forall \mathbf{h}, \widehat{\mathbf{h}} \in \mathbb{R}^d,$$

and therefore $\omega(\cdot;u) \in \mathcal{C}(\mathbb{R}^d)$.

Writing

$$(u * \varphi_\varepsilon)(\mathbf{x}) - u(\mathbf{x}) = \int_{\mathbb{R}^d} \varphi(\mathbf{z})(u(\mathbf{x} - \varepsilon\,\mathbf{z}) - u(\mathbf{x}))\mathrm{d}\mathbf{z},$$

we can easily bound

$$\begin{aligned} \|u * \varphi_\varepsilon - u\|^2_{\mathbb{R}^d} &\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \varphi(\mathbf{z})\mathrm{d}\mathbf{z} \right) \left(\int_{\mathbb{R}^d} \varphi(\mathbf{z}) |u(\mathbf{x} - \varepsilon\mathbf{z}) - u(\mathbf{x})|^2 \mathrm{d}\mathbf{z} \right) \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^d} \varphi(\mathbf{z}) \omega(\varepsilon\mathbf{z}; u)\mathrm{d}\mathbf{z}, \end{aligned}$$

and using the continuity of $\omega(\cdot;u)$ we can take the limit and finish the proof. $\square$

5 Sobolev spaces and Lipschitz transformations

Prerequesites and notes. Basic understanding of the distributional definition of $H^1(\Omega)$. The density of $\mathcal{D}(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$ will be used. Some basic tools on partitions of unity, such as the results of Section 1. The definition of support of an integrable function is also implicit to one of the arguments. I am not entirely sure where I took the ideas for this proof.

The aim of this very technical section is the proof of just one result that will allow us to apply Lipschitz change of variables in H^1 spaces. We start with a bijective map between two open sets

$$F : \Omega \to \mathcal{O} \qquad F^{-1} : \mathcal{O} \to \Omega.$$

We assume that F and F^{-1} are Lipschitz. Therefore, there exist two constants such that

$$C_1|\mathbf{x}_1 - \mathbf{x}_2| \le |F(\mathbf{x}_1) - F(\mathbf{x}_2)| \le C_2|\mathbf{x}_1 - \mathbf{x}_2| \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in \Omega.$$

In particular, F and F^{-1} are differentiable almost everywhere and their partial derivatives are bounded functions. For the reader who is not comfortable with Lipschitz maps, it suffices to think of a continuous piecewise $\mathcal{C}^1$ map, where the gradients have jump discontinuities in the boundaries of a simple partition of the set. Throughout this section, we will employ the symbol for the gradient vector to provide *a row vector* with the partial derivatives of a function. On the other hand $\partial_{x_j} F$ will be considered as a column vector and DF will be the matrix whose columns are the vectors $\partial_{x_j} F$.

The remainder of this section is devoted to proving, step by step, the following result

> Let $u \in H^1(\mathcal{O})$ and $F : \Omega \to \mathcal{O}$ a Lipschitz bijection with Lipschitz inverse. Then
>
> $$\partial_{x_j}(u \circ F) = (\nabla u \circ F)\partial_{x_j} F \qquad \forall j.$$
>
> As a consequence $u \circ F \in H^1(\Omega)$.

Without any further reference to these facts, we will assume that the *bounded open* sets Ω and $\mathcal{O}$ and the Lipschitz map F are the ones of the statement.

Lemma 12. *Let $\mathcal{O}$ be bounded. Assume that $u \in H^1(\mathcal{O})$ vanishes in a neighborhood of $\partial\mathcal{O}$. Then the function*

$$\widetilde{u} := \begin{cases} u & \text{in } \mathcal{O} \\ 0 & \text{in } \mathbb{R}^d \setminus \mathcal{O} \end{cases}$$

is in $H^1(\mathbb{R}^d)$ and $\nabla\widetilde{u} = \widetilde{\nabla u}$.

Proof. It is clear that $\widetilde{u} \in L^2(\mathbb{R}^d)$ and that the result is proved if we show that

$$\partial_{x_j}\widetilde{u} = \widetilde{\partial_{x_j} u} \qquad \forall j.$$

(The distributional derivative in the left-hand side is taken in $\mathbb{R}^d$, while the one in the righ-hand side is taken in $\mathcal{O}$.)

Using the techniques of Section 1 we can build $\phi \in \mathcal{D}(\Omega)$ such that $u\,\phi \equiv u$ and $\phi\,\partial_{x_j} u \equiv u$, i.e., $\phi \equiv 1$ where $u \neq 0$. Let then $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and note that

$$\begin{aligned}
\langle \partial_{x_j}\widetilde{u}, \varphi\rangle &= -\langle \widetilde{u}, \partial_{x_j}\varphi\rangle && \text{(definition of derivative)} \\
&= -\int_{\mathbb{R}^d} \widetilde{u}\,\partial_{x_i}\varphi = -\int_\Omega u\,\partial_{x_j}\varphi && (\widetilde{u} \text{ is regular}) \\
&= -\int_\Omega u\,\phi\,\partial_{x_j}\varphi && (u\,\phi = u) \\
&= -\int_\Omega u\partial_{x_j}(\phi\,\varphi) + \int_\Omega u\,(\partial_{x_j}\phi)\,\varphi && \\
&= -\langle u, \partial_{x_j}(\phi\,\varphi)\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} && (u\,\partial_{x_j}\phi \equiv 0,\ \phi\,\varphi \in \mathcal{D}(\Omega)) \\
&= \langle \partial_{x_j} u, \phi\,\varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} && \text{(definition of derivative)} \\
&= \int_\Omega (\partial_{x_j} u)\,\phi\,\varphi && (\partial_{x_j} u \text{ is regular}) \\
&= \int_\Omega (\partial_{x_j} u)\,\varphi && (\phi\,\partial_{x_j} u \equiv \partial_{x_j} u) \\
&= \int_{\mathbb{R}^d} \widetilde{\partial_{x_j} u}\,\varphi = \langle \widetilde{\partial_{x_j} u}, \varphi\rangle. && (\widetilde{\partial_{x_j} u} \text{ is regular})
\end{aligned}$$

This finishes the proof. □

Let us introduce the set of functions of $H^1(\mathcal{O})$ that can be approximated by functions in $\mathcal{C}^\infty(\overline{\mathcal{O}})$:

$$E(\mathcal{O}) := \Big\{ u \in H^1(\mathcal{O}) \ : \ \begin{array}{l} \exists (u_n) \subset \mathcal{C}^\infty(\overline{\mathcal{O}}) \\ u_n \xrightarrow{n\to\infty} u \,\text{in } H^1(\mathcal{O}) \end{array} \Big\}.$$

It is true that for many domains $E(\mathcal{O}) = H^1(\mathcal{O})$. This is however not a property that is satisfied by every domain. The functions of $E(\mathcal{O})$ are, intuitively, the functions that can be *extended* to functions of $H^1(\mathbb{R}^d)$.

Proposition 13. *If $u \in H^1(\mathcal{O})$ and $u \equiv 0$ near $\partial\mathcal{O}$, then $u \in E(\mathcal{O})$.*

Proof. By Lemma 12 the function $\widetilde{u}$ obtained by extension of u by zero to $\mathbb{R}^d \setminus \mathcal{O}$ is in $H^1(\mathbb{R}^d)$. Since $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$, $\widetilde{u}$ can be approximated by elements of $\mathcal{D}(\mathbb{R}^d)$. Restricting back to $\mathcal{O}$, the result is proved. □

Proposition 14. *If $u \in H^1(\mathcal{O})$ and $v \in E(\mathcal{O})$, then*

$$\nabla(u\,v) = u\nabla v + v\nabla u.$$

Proof. Given $v \in E(\mathcal{O})$, we consider a sequence $(v_n) \subset \mathcal{C}^\infty(\overline{\mathcal{O}})$ such that

$$v_n \stackrel{n\to\infty}{\longrightarrow} v \qquad \partial_{x_j} v_n \stackrel{n\to\infty}{\longrightarrow} \partial_{x_j} v \qquad \forall j \qquad \text{in } L^2(\mathcal{O}). \tag{7}$$

Note that

$$\nabla(u\,v_n) = u\nabla v_n + v_n\nabla u. \tag{8}$$

because $v_n \in \mathcal{C}^\infty(\overline{\mathcal{O}}) \subset \mathcal{C}^\infty(\mathcal{O})$. Also, (7) implies that

$$u\,v_n \stackrel{n\to\infty}{\longrightarrow} u\,v \qquad \text{in } \mathcal{D}'(\mathcal{O}). \tag{9}$$

(Convergence in $L^2(\mathcal{O})$ is converted in convergence in $L^1(\mathcal{O})$ after multiplication by fixed functions in $L^2(\mathcal{O})$ and convergence in $L^1(\mathcal{O})$ implies convergence in the sense of distributions.) Therefore

$$\partial_{x_j}(u\,v_n) \stackrel{n\to\infty}{\longrightarrow} u\,v \qquad \text{in } \mathcal{D}'(\mathcal{O}),$$

but at the same time, (7) and (8) imply that

$$\partial_{x_j}(u\,v_n) = u\,\partial_{x_j} v_n + v_n \partial_{x_j} u \stackrel{n\to\infty}{\longrightarrow} u\partial_{x_j} v + v\,\partial_{x_j} u \quad \forall j \qquad \text{in } \mathcal{D}'(\mathcal{O}).$$

This finishes the proof. □

Remark. Proposition 14 says that when $E(\mathcal{O}) = H^1(\mathcal{O})$, the product of two functions in $H^1(\mathcal{O})$ is in $L^1(\mathcal{O})$ with derivatives in $L^1(\mathcal{O})$.

Proposition 15. *Let $u \in H^1(\mathcal{O})$ and*

$$v \in \mathcal{C}(\mathcal{O}), \qquad \operatorname{supp} v \subset \mathcal{O}, \qquad \nabla v \in L^\infty(\mathcal{O})^d.$$

Then $u\,v \in E(\mathcal{O})$.

Proof. First of all, the properties of v imply that $v \in E(\mathcal{O})$ by Proposition 13. Note also that $u\,v \in L^2(\mathcal{O})$ because $v \in L^\infty(\mathcal{O})$. Proposition 14 implies that we can apply Leibniz' rule

$$\nabla(u\,v) = u\nabla v + v\nabla u \in L^2(\mathcal{O})^d,$$

so $u\,v \in H^1(\mathcal{O})$. Finally, we can apply again Proposition 13, because $u\,v \equiv 0$ in a neighborhood of $\partial\mathcal{O}$ (since v satisfies this property). □

Proposition 16. *If $u \in E(\mathcal{O})$, then*

$$\partial_{x_j}(u \circ F) = (\nabla u \circ F)\partial_{x_j} F \qquad \forall j. \tag{10}$$

Therefore $u \circ F \in H^1(\Omega)$.

Proof. The proof of this result is very similar to that of Proposition 14. We start by choosing a sequence $(u_n) \subset \mathcal{C}^\infty(\overline{\mathcal{O}})$ such that

$$u_n \overset{n\to\infty}{\longrightarrow} u \qquad \nabla u_n \overset{n\to\infty}{\longrightarrow} \nabla u \qquad \forall j \qquad \text{in } L^2(\mathcal{O}).$$

This implies that

$$u_n \circ F \overset{n\to\infty}{\longrightarrow} u \circ F \qquad (\nabla u_n) \circ F \overset{n\to\infty}{\longrightarrow} (\nabla u) \circ F \qquad \forall j \qquad \text{in } L^2(\Omega).$$

and finally

$$\partial_{x_j}(u \circ F) \overset{n\to\infty}{\longleftarrow} \partial_{x_j}(u_n \circ F) = (\nabla u_n \circ F)\partial_{x_j} F \overset{n\to\infty}{\longrightarrow} (\nabla u \circ F)\partial_{x_j} F,$$

with convergence in the sense of distributions. We have applied that (10) holds true for $u \in \mathcal{C}^\infty(\overline{\mathcal{O}})$ and that $\partial_{x_j} F \in L^\infty(\Omega)^d$. □

Lemma 17. *Let $u \in H^1(\mathcal{O})$ be such that $u \equiv 0$ in a neighborhood of $\partial\mathcal{O}$. Then*

$$\int_{\mathcal{O}} \partial_{x_j} u(\mathbf{x}) \mathrm{d}\mathbf{x} = 0 \qquad \forall j.$$

Proof. We first choose $\varphi \in \mathcal{D}(\mathcal{O})$ such that $\varphi \equiv 1$ in the support of u. Then

$$\int_{\mathcal{O}} \partial_{x_j} u = \int_{\mathcal{O}} \varphi\, \partial_{x_j} u = \langle \partial_{x_j} u, \varphi \rangle_{\mathcal{D}'(\mathcal{O})\times\mathcal{D}(\mathcal{O})} = -\langle u, \partial_{x_j}\varphi \rangle_{\mathcal{D}'(\mathcal{O})\times\mathcal{D}(\mathcal{O})} = -\int_{\mathcal{O}} u\, \partial_{x_j}\varphi = 0,$$

because $u\,\partial_{x_j}\varphi \equiv 0$ by construction of φ. □

Proposition 18. *Let $u \in H^1(\mathcal{O})$ and $F : \Omega \to \mathcal{O}$ a Lipschitz bijection with Lipschitz inverse. Then*

$$\partial_{x_j}(u \circ F) = (\nabla u \circ F)\partial_{x_j} F \qquad \forall j.$$

As a consequence $u \circ F \in H^1(\Omega)$.

Proof. Let $\varphi \in \mathcal{D}(\Omega)$. Then the function $\varphi \circ F^{-1}$ is continuous is $\overline{\mathcal{O}}$, has compact support in $\mathcal{O}$, and has bounded gradient. Therefore, by Propositions 15 and 16

$$w := u\,(\varphi \circ F^{-1}) \in E(\mathcal{O}), \qquad \text{and} \qquad w \circ F \in H^1(\Omega).$$

Also, by Proposition 14

$$\nabla w = (\varphi \circ F^{-1})\nabla u + u\nabla(\varphi \circ F^{-1}) = (\varphi \circ F^{-1})\nabla u + u(\nabla\varphi \circ F^{-1})DF^{-1}. \tag{11}$$

Using Proposition 16 we prove that

$$\begin{aligned}
\partial_{x_j}(w \circ F) &= (\nabla w \circ F)\partial_{x_j} F \\
&= \varphi(\nabla u \circ F)\partial_{x_j} F + (u \circ F)\nabla\varphi \underbrace{(DF^{-1} \circ F)\partial_{x_j} F}_{(DF)^{-1}\partial_{x_j} F = \mathbf{e}_j} \quad \text{(by (11))} \\
&= \varphi(\nabla u \circ F)\partial_{x_j} F + (u \circ F)\partial_{x_j}\varphi.
\end{aligned}$$

Since $w \circ F = (u \circ F)\,\varphi \in H^1(\Omega)$ vanishes in a neighborhood of $\partial\Omega$, then by Lemma 17

$$\begin{aligned} 0 = \int_\Omega \partial_{x_j}(w \circ F) &= \int_\Omega (\nabla u \circ F)\partial_{x_j} F\,\varphi + \int_\Omega (u \circ F)\,\partial_{x_j}\varphi \\ &= \langle (\nabla u \circ F)\partial_{x_j} F, \varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} + \langle u \circ F, \partial_{x_j}\varphi\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}. \end{aligned}$$

Since this equality holds for all $\varphi \in \mathcal{D}(\Omega)$, this proves that

$$(\nabla u \circ F)\partial_{x_j} F = \partial_{x_j}(u \circ F)$$

as distributions in Ω. Finally $u \circ F \in L^2(\Omega)$, $\nabla u \circ F \in L^2(\Omega)^d$ and $\partial_{x_j} F \in L^\infty(\Omega)^d$, which proves that $u \circ F \in H^1(\Omega)$. $\square$

Contents

References

[1] Lars Hörmander, *The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis.* Reprint of the second (1990) edition. Classics in Mathematics. Springer-Verlag, Berlin, 2003.