MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Take home exam # 1 Due February 22

Instructions. Write problems on separate pages. Write down all details. Proofs should be thoroughly explained with no holes and overstatements.

1. (20 points) Distributions Worksheet. Problem 6.

2. (20 points) Distributions Worksheet. Problem 8.

3. (20 points) Distributions Worksheet. Problem 10.

4. (40 points) Distributions Worksheet. Problem 12. (The fundamental solution for the Helmholtz equation.)

5. (30 points) Homogeneous Dirichlet Problem Worksheet. Problem 1. (The space $H^2(\Omega)$.)

6. (40 points) Let Ω be a bounded domain and $u \in L^1(\Omega)$ be such that $u \equiv 0$ in a neighborhood of $\partial\Omega$ and $\partial_{x_i} u \in L^1(\Omega)$. Let $\widetilde{u}, \widetilde{\partial_{x_i} u} : \mathbb{R}^d \to \mathbb{R}$ be given by

$$\widetilde{u} := \left\{ \begin{array}{ll} u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{array} \right. \qquad \widetilde{\partial_{x_i} u} := \left\{ \begin{array}{ll} \partial_{x_i} u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{array} \right.$$

Show that

$$\partial_{x_i} \widetilde{u} = \widetilde{\partial_{x_i} u}.$$

7. (30 points) Let $u \in H^1(\mathbb{R}^d)$ satisfy

$$-\Delta u + u = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Show that $u = 0$. (**Hint.** Show that u is orthogonal in $H^1(\mathbb{R}^d)$ to all $\mathcal{D}(\mathbb{R}^d)$ functions.)