MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Take home exam # 2 Due April 8

Instructions. Type the solutions in LaTeX. Write the problems in consecutive order. Write down all details. Proofs should be thoroughly explained with no holes and overstatements.

1. (20 points) Chapter III worksheet. Problem 2(d)
2. (30 points) Chapter III worksheet. Problem 7.
3. (20 points) Chapter IV worksheet. Problem 1.
4. (30 points) Chapter IV worksheet. Problem 6.
5. (30 points) Let Ω be a bounded open domain with the H^1 extension property and let B be another bounded open domain such that $\overline{\Omega} \subset B$. Show that there exists a bounded extension operator $H^1(\Omega) \to H^1_0(B)$.
6. (20 points) Let $\mathbb{R}^d_+ := \{\mathbf{x} \in \mathbb{R}^d \,:\, x_d > 0\}$. Show that the set
$$\{u \in \mathcal{C}^\infty(\overline{\mathbb{R}^d_+}) \,:\, \operatorname{supp} u \text{ is bounded}\}$$
(the support of u is compact but can reach the boundary) is dense in $H^1(\mathbb{R}^d_+)$.
7. (50 points) **Fourier characterization of $H^1(\mathbb{R}^d)$.** Given $u \in \mathcal{D}(\mathbb{R}^d)$, we define $\widehat{u} : \mathbb{R}^d \to \mathbb{C}$ by
$$\widehat{u}(\boldsymbol{\xi}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} u(\mathbf{x}) e^{\imath\, \mathbf{x}\cdot\boldsymbol{\xi}} \mathrm{d}\mathbf{x}.$$
   (a) Show that $\widehat{u} \in L^2_{\mathbb{C}}(\mathbb{R}^d)$ and
$$\|\widehat{u}\|_{\mathbb{R}^d} = \|u\|_{\mathbb{R}^d}.$$
   (b) Show that there exists a unique linear isometry $\mathcal{F} : L^2_{\mathbb{C}}(\mathbb{R}^d) \to L^2_{\mathbb{C}}(\mathbb{R}^d)$ such that $\mathcal{F}(u) = \widehat{u}$ for all $u \in \mathcal{D}(\mathbb{R}^d)$.

   (c) Show that if $u \in H^1_{\mathbb{C}}(\mathbb{R}^d)$, and we consider the function $m_j(\boldsymbol{\xi}) := \imath \xi_j$, then
$$\mathcal{F}(\partial_{x_j} u) = m_j\, \mathcal{F}(u).$$
   (d) Show that if $u \in H^1_{\mathbb{C}}(\mathbb{R}^d)$, then
$$\|u\|^2_{H^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2)\, |\mathcal{F}(u)(\boldsymbol{\xi})|^2\, \mathrm{d}\boldsymbol{\xi}.$$
   (e) Let $u \in L^2_{\mathbb{C}}(\mathbb{R}^d)$. Show that $u \in H^1_{\mathbb{C}}(\mathbb{R}^d)$ if and only if
$$\int_{\mathbb{R}^d} (1 + |\boldsymbol{\xi}|^2) |\mathcal{F}(u)(\boldsymbol{\xi})|^2 \mathrm{d}\boldsymbol{\xi} < \infty.$$