MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Problems I. Distributions

1. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be such that
$$\int_{\mathbb{R}^d} \varphi(\mathbf{x})\mathrm{d}\mathbf{x} = 1.$$
Let then
$$\varphi_\varepsilon(\mathbf{x}) := \tfrac{1}{\varepsilon^d}\varphi(\tfrac{1}{\varepsilon}\,\mathbf{x}).$$
Show that for all $f \in \mathcal{C}(\mathbb{R}^d)$,
$$\int_{\mathbb{R}^d} \varphi_\varepsilon(\mathbf{x}) f(\mathbf{y}-\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d} \varphi_\varepsilon(\mathbf{y}-\mathbf{x}) f(\mathbf{x})\mathrm{d}\mathbf{x} \stackrel{\varepsilon\to 0}{\longrightarrow} f(\mathbf{y}) \qquad \forall \mathbf{y} \in \mathbb{R}^d.$$

2. Let $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ in $\mathbb{R}^d$ and let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Show that
$$\varphi(\,\cdot\, - \mathbf{x}_n) \stackrel{n\to\infty}{\longrightarrow} \varphi(\,\cdot\, - \mathbf{x}) \qquad \text{in } \mathcal{D}(\mathbb{R}^d).$$

3. Let $\mathbf{h} \in \mathbb{R}^d$, (c_n) be a sequence of real numbers such that $\lim_{n\to\infty} c_n = 0$, and let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Show that
$$\frac{1}{|c_n|}(\varphi(\,\cdot\, + c_n\mathbf{h}) - \varphi) \stackrel{n\to\infty}{\longrightarrow} \mathbf{h}\cdot\nabla\varphi \qquad \text{in } \mathcal{D}(\mathbb{R}^d).$$

4. Let $\phi \in \mathcal{C}^\infty(\Omega)$, Show that
$$\varphi_n \stackrel{n\to\infty}{\longrightarrow} \varphi \quad \text{in } \mathcal{D}(\Omega) \qquad \Longrightarrow \qquad \phi\,\varphi_n \stackrel{n\to\infty}{\longrightarrow} \phi\,\varphi \quad \text{in } \mathcal{D}(\Omega),$$
that is, multiplication by a $\mathcal{C}^\infty(\Omega)$ function is a sequentially continuous operator in $\mathcal{D}(\Omega)$.

5. Show that in the sense of distributions $\partial^{\alpha+\beta} = \partial^\alpha\partial^\beta$ for every pair of multi-indices.

6. (Salsa, Problem 7.5) Let $u(x) := |x|$. Compute u' and u'' in the sense of distributions on $\mathbb{R}$.

7. (Salsa, Problem 7.3) Let (c_k) be a sequence of real numbers such that there exists $p \geq 0$ satisfying
$$|c_k| \leq C\,k^p \qquad \forall k.$$
Show that the series
$$\sum_{k=1}^\infty c_k \sin(k\,\cdot\,)$$
converges in the sense of distributions in $\mathcal{D}'(\mathbb{R})$. **Hint.** Show that
$$\Big|\int_{-\infty}^\infty \varphi(x)\sin(k\,x)\mathrm{d}x\Big| \leq C_m(\varphi)\,k^{-m} \qquad \forall k \geq 0, \qquad \forall m.$$

8. Show that $\partial^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is sequentially continuous.

9. (**Sequences converging to a Dirac delta**) Let $\xi \in L^1(\mathbb{R}^d)$ be such that

$$\int_{\mathbb{R}^d} \xi(\mathbf{x}) \mathrm{d}\mathbf{x} = 1.$$

Consider then the functions

$$\xi_\varepsilon := \tfrac{1}{\varepsilon^d} \xi(\tfrac{1}{\varepsilon} \cdot).$$

Show that

$$\xi_\varepsilon \stackrel{\varepsilon \to 0}{\longrightarrow} \delta_{\mathbf{0}} \qquad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

10. Let $f_n \to f$ in $L^2(\Omega)$. Show that $f_n \to f$ in $\mathcal{D}'(\Omega)$.

11. (**Multiplication of distributions by $\mathcal{C}^\infty$ functions**) Let $T \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{C}^\infty(\Omega)$. We define $\phi T : \mathcal{D}(\Omega) \to \mathbb{R}$ with the formula

$$\langle \phi T, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} := \langle T, \phi \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}.$$

Show that $\phi T \in \mathcal{D}'(\Omega)$.

12. (**Fundamental solution for the Helmholtz equation**) Consider the function $\Phi : \mathbb{R}^3 \to \mathbb{R}$, given by

$$\Phi(\mathbf{x}) := \frac{e^{-\imath k |\mathbf{x}|}}{4\pi |\mathbf{x}|}.$$

Show that

$$\Delta \Phi + k^2 \Phi = -\delta_{\mathbf{0}} \qquad \text{in } \mathcal{D}'(\mathbb{R}^3).$$