MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Problems II. The homogeneous Dirichlet problem

1. Consider the space

$$\begin{aligned} H^2(\Omega) &:= \{u \in L^2(\Omega) \,:\, \partial^\alpha u \in L^2(\Omega) \quad \forall \alpha \in \mathbb{Z}^d, |\alpha| \le 2\} \\ &= \{u \in H^1(\Omega) \,:\, \nabla u \in H^1(\Omega)^d\}, \end{aligned}$$

endowed with the norm

$$\begin{aligned} \|u\|^2_{H^2(\Omega)} &:= \|u\|^2_{H^1(\Omega)} + \sum_{|\alpha|=2} \|\partial^\alpha u\|^2_\Omega \\ &= \|u\|^2_\Omega + \|\nabla u\|^2_\Omega + \|\mathrm{D}^2 u\|^2_\Omega, \end{aligned}$$

where $\mathrm{D}^2 u$ is a vector containing the $d(d+1)/2$ second partial derivatives of u.

   (a) Show that $H^2(\Omega)$ is a Hilbert space. (Note that this includes finding an inner product whose associated norm is the one we have given.)

   (b) Show that the inclusion $I : H^2(\Omega) \to H^1(\Omega)$ is a continuous operator.

   (c) Show that $\partial_{x_i} : H^2(\Omega) \to H^1(\Omega)$ is a continuous operator.

2. Let Ω be a bounded open set such that the measure of $\partial\Omega$ is zero. Show that $\chi_\Omega \notin H^1(\mathbb{R}^d)$. (**Hint.** Show that the partial derivatives are not regular with the same technique we used for the Dirac delta.)

3. **Several views of $H^1_0(\Omega)$.**

   (a) Using mollification, show that

$$\{u \in H^1(\Omega) \,:\, u \equiv 0 \text{ near } \partial\Omega\} \subset H^1_0(\Omega),$$

   where by $u \equiv 0$ near $\partial\Omega$ we mean that there exists $\varepsilon > 0$ such that $u \equiv 0$ in the set $\{\mathbf{x} \in \Omega \,:\, \mathrm{dist}(\mathbf{x}, \partial\Omega) < \varepsilon\}$. Show that the space on the left is dense in $H^1_0(\Omega)$.

   (b) Consider the space

$$\mathcal{C}^1_{00}(\Omega) := \{u \in \mathcal{C}^1(\overline{\Omega}) \,:\, \mathrm{supp}\, u \text{ compact in } \Omega\}.$$

   Show that $\mathcal{C}^1_{00}(\Omega)$ is dense in $H^1_0(\Omega)$.

   (c) Let $v \in \mathcal{C}^1(\overline{\Omega})$. Show that $u \mapsto v\,u$ maps $H^1_0(\Omega)$ into itself. (**Hint.** Note that we are not demanding $v \in \mathcal{C}^\infty(\Omega)$, so in principle, it is not clear whether we can asser that $v\,u \in H^1(\Omega)$. You will need to use a density argument.)

4. **Reaction-difusion problems.** Let Ω be a bounde domain, let $\kappa, c \in L^\infty(\Omega)$ be such that

$$\kappa(\mathbf{x}) \ge \kappa_0 > 0, \qquad c(\mathbf{x}) \ge 0, \qquad \text{almost everywhere},$$

and let $f \in L^2(\Omega)$. Consider now the problem

$$u \in H^1_0(\Omega) \qquad -\operatorname{div}(\kappa \nabla u) + c\,u = f,$$

with all the differential operators understood in the sense of distributions.

(a) Find an equivalent variational formulation for this problem.

(b) Write the equivalent minimization problem.

(c) Show existence and uniqueness of solution of the problem.

(d) Find a constant (independent of f) so that

$$\|u\|_{H^1(\Omega)} \leq C_{\text{pb}} \|f\|_\Omega.$$

How does this constant depend on the coefficients of the equation (κ and c) and on the domain?

5. **The clamped Kirchhoff plate.** Consider the space (see Problem 1)

$$H_0^2(\Omega) := \left\{ u \in H^2(\Omega) \ : \ \begin{array}{l} \exists (\varphi_n) \subset \mathcal{D}(\Omega) \\ \varphi_n \to u \quad \text{in } H^2(\Omega) \end{array} \right\}$$

(a) Show that if $u \in H_0^2(\Omega)$, then $u, \partial_{x_i} u \in H_0^1(\Omega)$.

(b) Use (a) to show that you can find $C_\Omega > 0$ such that

$$\|u\|_{H^2(\Omega)} \leq C_\Omega \|\mathrm{D}^2 u\|_\Omega.$$

(c) Using a density argument and differentiation in the sense of distributions, show that

$$(\partial_{x_i}\partial_{x_j} u, \partial_{x_i}\partial_{x_j} u)_\Omega = (\partial_{x_i}^2 u, \partial_{x_j}^2 u)_\Omega \qquad \forall u \in H_0^2(\Omega).$$

(d) Use (b) and (c) to show that

$$\|\Delta u\|_\Omega$$

defines a norm in $H_0^2(\Omega)$ that is equivalent to the usual one.

(e) Finally, for given $f \in L^2(\Omega)$, consider the problem

$$u \in H_0^2(\Omega) \qquad \Delta^2 u = f.$$

Write equivalent variational formulations and minimization principles. Show existence and uniqueness of solution.