MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Problems III. The non-homogeneous Dirichlet problem

1. If Ω is a Lipschitz domain, show that $\mathbb{R}^d \setminus \overline{\Omega}$ also satisfies the H^1-extension property.

2. **The symmetry argument.** Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth version of the Heaviside function

$$h \in \mathcal{C}^\infty(\mathbb{R}), \quad 0 \le h \le 1, \quad \operatorname{supp} h = [0,\infty), \quad \operatorname{supp}(1-h) = (-\infty, 1]$$

and let $h_n(\mathbf{x}) := h(n\, x_d - 1)$. We will write

$$\mathbb{R}^d \ni \mathbf{x} = (\widetilde{\mathbf{x}}, x_d) \longmapsto \check{\mathbf{x}} := (\widetilde{\mathbf{x}}, -x_d)$$

and consider the extension operator for functions $u : \mathbb{R}^d_+ \to \mathbb{R}$,

$$(Eu)(\mathbf{x}) := \begin{cases} u(\mathbf{x}), & \text{if } \mathbf{x} \in \mathbb{R}^d_+, \\ u(\check{\mathbf{x}}), & \text{if } x_d < 0. \end{cases}$$

(a) Make a plot of the functions h_n and show that

$$h_n\psi \in \mathcal{D}(\mathbb{R}^d_+), \qquad h_n\psi \to \psi \text{ in } L^2(\mathbb{R}^d_+) \quad \forall \psi \in \mathcal{D}(\mathbb{R}^d).$$

(b) Show that if $u \in L^2(\mathbb{R}^d_+)$, then

$$\langle Eu, \psi \rangle = \int_{\mathbb{R}^d_+} u(\mathbf{x}) \big(\psi(\mathbf{x}) + \psi(\check{\mathbf{x}})\big) \mathrm{d}\mathbf{x} \qquad \forall \psi \in \mathcal{D}(\mathbb{R}^d).$$

(c) By carefully playing with the functions h_n, show that if $u \in H^1(\mathbb{R}^d_+)$, then

$$\partial_{x_j}(Eu) = E(\partial_{x_j} u) \qquad 1 \le j \le d-1.$$

(d) Show that

$$(\partial_{x_d} h_n)(\varphi - \varphi(\check{\cdot})) \to 0 \text{ in } L^2(\mathbb{R}^d_+) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

(e) Finally, using (a) and (d), show that if $u \in H^1(\mathbb{R}^d_+)$, then

$$\langle \partial_{x_d}(Eu), \varphi \rangle = \int_{\mathbb{R}^d} \partial_{x_d} u(\mathbf{x}) \big(\varphi(\mathbf{x}) - \varphi(\check{\mathbf{x}})\big) \mathrm{d}\mathbf{x} \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

The previous results show that if $u \in H^1(\mathbb{R}^d)$, then $Eu \in H^1(\mathbb{R}^d)$. Why?

3. **An extension operator for $H^2(\mathbb{R}^d_+)$.** Given $u \in H^2(\mathbb{R}^d_+)$, we define

$$(Eu)(\mathbf{x}) = (Eu)(\widetilde{\mathbf{x}}, x_d) := \begin{cases} u(\widetilde{\mathbf{x}}, x_d) & \text{if } x_d > 0, \\ 4u(\widetilde{\mathbf{x}}, -\tfrac{1}{2}x_d) - 3u(\widetilde{\mathbf{x}}, -\tfrac{1}{3}x_d), & \text{if } x_d < 0. \end{cases}$$

(a) Show that $Eu \in H^2(\mathbb{R}^d)$.

(b) Show that $\|Eu\|_{\mathbb{R}^d} \le C_0 \|u\|_{\mathbb{R}^d_+}$ for all $u \in L^2(\mathbb{R}^d_+)$.

(c) Show that $\|Eu\|_{H^1(\mathbb{R}^d)} \le C_1 \|u\|_{H^1(\mathbb{R}^d_+)}$ for all $u \in H^1(\mathbb{R}^d_+)$.

(d) Show that $\|Eu\|_{H^2(\mathbb{R}^d)} \le C_2 \|u\|_{H^2(\mathbb{R}^d_+)}$ for all $u \in H^2(\mathbb{R}^d_+)$.

4. **Understanding $H^{1/2}(\Gamma)$.**

   (a) Assume that $\partial\Omega$ is composed of two disjoint connected parts, Γ_1 and Γ_2, each of them the boundary of a Lipschitz domain (think of an annular domain). Show that

   $$H^{1/2}(\Gamma) \equiv H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2).$$

   (**Hint.** Use $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi_1 + \varphi_2 \equiv 1$ in a neighborhood of Ω and such that

   $$\operatorname{supp}\varphi_2 \cap \Gamma_1 = \emptyset \quad \text{and} \qquad \operatorname{supp}\varphi_2 \cap \Gamma_1 = \emptyset,$$

   to separate the boundaries.)

   (b) Let Ω be a Lipschitz domain and $\Gamma_{\mathrm{pc}} \subset \partial\Omega$ a subset of its boundary such that it is possible to integrate on it. Consider the operator $\gamma_{\mathrm{pc}} : H^1(\Omega) \to L^2(\Gamma_{\mathrm{pc}})$ given by

   $$\gamma_{\mathrm{pc}} u := (\gamma u)|_{\Gamma_{\mathrm{pc}}}.$$

   Show that this operator is the only possible extension of the operator

   $$\begin{array}{rcl} H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}) & \longrightarrow & L^2(\Gamma_{\mathrm{pc}}) \\ u & \longmapsto & u|_{\Gamma_{\mathrm{pc}}}. \end{array}$$

   (Note that the restriction operators in the previous formulas are different to each other. Why?)

5. **The trace from an exterior domain.** Let Ω_- be a bounded Lipschitz domain and $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$. Since both $\Omega_\pm$ satisfy the extension property, we can define different trace operators

   $$\gamma^\pm : H^1(\Omega_\pm) \to L^2(\Gamma).$$

   (a) Show that if $u \in H^1(\mathbb{R}^d)$, then $\gamma^+ u = \gamma^- u$.

   (b) Show that the range of both trace operators is the same.

   (c) Show that if $u \in H^1(\mathbb{R}^d \setminus \Gamma)$ and $\gamma^+ u = \gamma^- u$, then $u \in H^1(\mathbb{R}^d)$. (**Hint.** Let $u_\pm := u|_{\Omega_\pm}$. Extend u_+ to an element of $H^1(\mathbb{R}^d)$ and show that this extension minus u_- is in $H^1_0(\Omega_-)$.)

6. **Reaction-diffusion problems.** On a bounded Lipschitz domain, we consider two coefficients

   $$\kappa, c \in L^\infty(\Omega), \qquad \kappa \ge \kappa_0 > 0, \qquad c \ge 0 \qquad \text{(almost everywhere)}$$

   and two data functions $(f, g) \in L^2(\Omega) \times H^{1/2}(\Gamma)$. Consider the problem

   $$\left[\begin{array}{ll} u \in H^1(\Omega), \quad \gamma u = g, & \\ -\operatorname{div}(\kappa \nabla u) + c\, u = f & \text{in } \Omega. \end{array}\right.$$

   (a) Write its equivalent variational formulation ad the associated minimization problem.

   (b) Show well-posedness of this problem.

7. **The optimal lifting.** Consider the operator $\gamma^\dagger : H^{1/2}(\Gamma) \to H^1(\Omega)$, given by $u = \gamma^\dagger g$ is the solution of

$$\left[\begin{array}{l} u \in H^1(\Omega), \quad \gamma u = g, \\ -\Delta u + u = 0 \quad \text{in } \Omega. \end{array}\right.$$

Show that it is well defined, linear, bounded. Write the associated minimization problem and show that $\gamma^\dagger$ is the Moore-Penrose pseudoinverse of the trace $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$.