MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Problems IV. Non-symmetric and complex problems

1. **Non-negative distributions.** Let $T \in \mathcal{D}'(\Omega)$. We say that $T \geq 0$ when

$$\langle T, \varphi \rangle \geq 0 \qquad \forall \varphi \in \mathcal{D}_+(\Omega) := \{\varphi \in \mathcal{D}(\Omega) \,:\, \varphi \geq 0\}$$

Show that this definition is coherent for regular distributions, that is, when $T = f \in L^1_{\mathrm{loc}}(\Omega)$, then $T \geq 0$ is equivalent to $f \geq 0$ (almost everywhere).

2. Let Ω be a bounded open set and $\boldsymbol{\kappa} : \Omega \to \mathbb{R}^{d\times d}$ be a matrix valued function satisfying:

$$\kappa_{ij} \in L^\infty(\Omega) \quad \forall i, j,$$

and

$$\sum_{i,j=1}^{d} \kappa_{ij}(\cdot)\xi_i\xi_j \geq \kappa_0 \sum_{j=1}^{d} |\xi_i|^2 \quad \text{almost everywhere} \qquad \forall (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d,$$

   (a) Study well-posedness of the problem

$$\left[\begin{array}{l} u \in H^1_0(\Omega), \\ (\boldsymbol{\kappa}\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H^1_0(\Omega). \end{array}\right.$$

   (b) Write an equivalent boundary value problem.

   (c) Show that the components of $\boldsymbol{\kappa}^{-1}$ are $L^\infty(\Omega)$ functions.

   (d) Show that is $\boldsymbol{\kappa}^\top = \boldsymbol{\kappa}$, then there is an associated minimization principle and the expression

$$\|u\|_\kappa := \int_\Omega (\boldsymbol{\kappa}\nabla u) \cdot \nabla u$$

   defines an equivalent norm in $H^1_0(\Omega)$.

3. Prove Riesz-Fréchet's Theorem in the complex case, namely, the map

$$\begin{array}{ccc} V & \longrightarrow & V^* \\ u & \longmapsto & (u, \cdot\,)_V \end{array}$$

is an isometric isomorphism between a complex Hilbert space V and its antidual V^*.

4. Let V be a complex vector space, let $a : V \times V \to \mathbb{C}$ be sesquilinear (linear in the first component, conjugate linear in the second one) and $\ell : V \to \mathbb{C}$ be conjugate linear. Show that the minimization problem

$$\tfrac{1}{2} a(u, u) - \ell(u) = \min!, \qquad u \in V$$

is equivalent to the variational problem

$$\left[\begin{array}{l} u \in V, \\ a(u, v) = \ell(v), \qquad \forall v \in V. \end{array}\right.$$

(**Hint.** Show that the following problem

$$\left[\begin{array}{l} u \in V, \\ \operatorname{Re} a(u,v) = \operatorname{Re} \ell(v), \qquad \forall v \in V, \end{array}\right.$$

is equivalent to both problems.)

5. Let V be a complex vector space endowed with a conjugate linear involution that we will call conjugation, that is, we have a map $V \to V$, whose action we denote $u \mapsto \overline{u}$ such that

$$\overline{\overline{u}} = u, \qquad \overline{u+v} = \overline{u} + \overline{v}, \qquad \overline{\alpha\, u} = \overline{\alpha}\,\overline{u}, \qquad \forall u, v \in V, \quad \forall \alpha \in \mathbb{C}.$$

(Note that we are using the overline symbol with two different meanings in the last formula.)

   (a) Show that there exists a real vector space W whose complexification is V. (**Hint.** Consider the space $W = \{u \in V \,:\, u = \overline{u}\}$ with multiplication by real scalars.)

   (b) Assume that V is an inner product space and that

$$\overline{(u,v)_V} = (\overline{u}, \overline{v})_V \qquad \forall u, v \in V.$$

   Show that we can endow W with an inner product so that, when we complexify, we recover the inner product of V.

6. Consider two functions $f_1, f_2 \in L^2(\Omega)$ and the following system of boundary value problems (here Ω is a bounded set):

$$\left[\begin{array}{l} u_1, u_2 \in H_0^1(\Omega), \\ -\Delta u_1 + u_2 = f_1, \\ \Delta u_2 + u_1 = f_2. \end{array}\right.$$

(Note the different signs of the Laplacians.)

   (a) Write an equivalent variational formulation working on the space $V = H_0^1(\Omega) \times H_0^1(\Omega)$:

$$\left[\begin{array}{l} (u_1, u_2) \in V, \\ a\big((u_1,u_2),(v_1,v_2)\big) = \ell\big((v_1,v_2)\big) \quad \forall (v_1,v_2) \in V. \end{array}\right.$$

   (b) Consider now the function $u = u_1 + \imath u_2 \in H_0^1(\Omega;\mathbb{C}) =: V_{\mathbb{C}}$. Write the BVP as a problem in the variable u, find its equivalent variational formulation and show that it is well posed.

   (c) The strategy used in (b) to show coercivity can be used to find a transformation $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$a\big((u_1,u_2), \mathrm{R}(u_1,u_2)\big) \geq \alpha \|(u_1,u_2)\|_V^2 \qquad \forall (u_1,u_2) \in V.$$