MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Problems V. Neumann boundary conditions

1. Let Ω be a bounded Lipschitz domain with boundary Γ. Consider the space

$$H^1_\Delta(\Omega) := \{u \in H^1(\Omega) \, : \, \Delta u \in L^2(\Omega)\},$$

endowed with the norm

$$\|u\|^2_{H^1_\Delta(\Omega)} := \|u\|^2_\Omega + \|\nabla u\|^2_\Omega + \|\Delta u\|^2_\Omega.$$

   (a) Show that it is a Hilbert space. (This includes finding the inner product.)
   (b) Show that $\nabla : H^1(\Omega) \to \mathbf{H}(\mathrm{div}, \Omega)$ is bounded.
   (c) Show that the normal derivative map $\partial_\nu : H^1_\Delta(\Omega) \to H^{-1/2}(\Gamma)$, given by $\partial_\nu u := (\nabla u) \cdot \mathbf{n}$ is bounded and surjective.

2. Let Ω be a bounded Lipschitz domain. Show that the divergence operator

$$\mathrm{div} : \mathbf{H}(\mathrm{div}, \Omega) \longrightarrow L^2(\Omega)$$

is surjective. (**Hint.** Solve a Laplacian and take a gradient.)

3. Let Ω be a bounded Lipschitz domain with boundary Γ. Let $\alpha \in L^\infty(\Gamma)$ be non-negative. Use the Deny-Lions theorem to show that the bilinear form

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Gamma \alpha \, (\gamma u)(\gamma v)$$

is coercive in $H^1(\Omega)$ if and only if $\alpha \neq 0$. (Note that we have assumed $\alpha \geq 0$.)

4. **A variant of the Deny-Lions theorem.** Let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be bilinear, bounded and such that

$$a(u, u) \geq C \|\nabla u\|^2_\Omega \quad \forall u \in H^1(\Omega), \qquad a(1, 1) \neq 0.$$

Show that a is coercive in $H^1(\Omega)$. (**Hint.** Assume that it is not coercive, and follow the proof of the Deny-Lions Theorem.)

5. **Convection-diffusion.** We want to find conditions on $\boldsymbol{\beta}$ and c ensuring that the bilinear form

$$(\nabla u, \nabla v)_\Omega + (\boldsymbol{\beta} \cdot \nabla u, v)_\Omega + (c\, u, v)_\Omega$$

is bounded and coercive in $H^1(\Omega)$.

   (a) Show that $c \in L^\infty(\Omega), \boldsymbol{\beta} \in \mathcal{C}^1(\overline{\Omega})^d$ satisfying

$$\int_\Omega (c - \tfrac{1}{2} \nabla \cdot \boldsymbol{\beta}) u^2 + \tfrac{1}{2} \int_\Gamma (\boldsymbol{\beta} \cdot \boldsymbol{n}) \, (\gamma u)^2 \geq 0 \qquad \text{and} \qquad \int_\Omega c > 0$$

   are sufficient conditions for coercivity.

(b) Show that $c \in L^\infty(\Omega), \boldsymbol{\beta} \in L^\infty(\Omega)^d$ satisfying $\nabla \cdot \boldsymbol{\beta} \in L^\infty(\Omega)$, $\boldsymbol{\beta} \cdot \mathbf{n} = 0$ (this normal trace is taken in the sense of $\mathbf{H}(\mathrm{div}, \Omega)$) and

$$c - \tfrac{1}{2}\nabla \cdot \boldsymbol{\beta} \geq 0, \qquad \text{and} \qquad \int_\Omega c > 0,$$

are also sufficient conditions for coercivity.

(**Hint.** At a crucial moment, the result of the previous exercise is quite useful.)

6. **Trace spaces on parts of the boundary.** Let Ω be a bounded Lipschitz domain with boundary Γ, let $\Gamma_D \subset \Gamma$ be a relatively open subset of the boundary with positive $(d-1)$-dimensional measure, and let $\Gamma_N := \Gamma \setminus \overline{\Gamma_D}$. Consider the space

$$V_D := \{u \in H^1(\Omega) \,:\, \gamma u = 0 \quad \text{on } \Gamma_D.\}$$

(a) Show that V_D is a closed subspace of $H^1(\Omega)$ and that $\|\nabla \cdot\|_\Omega$ defines an equivalent norm in V_D.

(b) Consider the space:

$$H^{1/2}(\Gamma_N) := \{\rho|_{\Gamma_N} \,:\, \rho \in H^{1/2}(\Gamma)\},$$

endowed with the image norm

$$\|\xi\|_{H^{1/2}(\Gamma_N)} := \inf\{\|\rho\|_{H^{1/2}(\Gamma)} \,:\, \rho|_{\Gamma_N} = \xi\}.$$

Show that this norm is equal to the norm

$$\|\xi\| := \inf\{\|u\|_{H^1(\Omega)} \,:\, \gamma u|_{\Gamma_N} = \xi\},$$

which is the image norm of the trace-and-restriction operator $H^1(\Omega) \to L^2(\Gamma_N)$. Show that there exists a bounded extension operator $H^{1/2}(\Gamma_N) \to H^{1/2}(\Gamma)$.

(c) Consider the space

$$\begin{aligned} \widetilde{H}^{1/2}(\Gamma_N) &:= \{\xi \in L^2(\Gamma_N) \,:\, \xi = \gamma u|_{\Gamma N}, \quad u \in V_D\} \\ &= \{\rho|_{\Gamma_N} \,:\, \rho \in H^{1/2}(\Gamma), \quad \rho|_{\Gamma_D} = 0\} \\ &= \{\xi \in H^{1/2}(\Gamma_N) \,:\, \widetilde{\xi} \in H^{1/2}(\Gamma)\}, \end{aligned}$$

where we have used the extension-by-zero operator

$$L^2(\Gamma_N) \ni \xi \longmapsto \widetilde{\xi} \in L^2(\Gamma), \qquad \widetilde{\xi} := \begin{cases} \xi & \text{in } \Gamma_N, \\ 0 & \text{in } \Gamma_D. \end{cases}$$

Show that all three definitions provide the same space. In this space we choose the norm

$$\|\xi\|_{\widetilde{H}^{1/2}(\Gamma_N)} := \|\widetilde{\xi}\|_{H^{1/2}(\Gamma)} = \inf\{\|u\|_{H^1(\Omega)} \,:\, \gamma u = \widetilde{\xi}\}.$$

Show that

$$\|\xi\|_{H^{1/2}(\Gamma_N)} \leq \|\xi\|_{\widetilde{H}^{1/2}(\Gamma_N)} \qquad \forall \xi \in \widetilde{H}^{1/2}(\Gamma_N).$$

(d) Show that

$$\widetilde{H}^{1/2}(\Gamma_N) \subset H^{1/2}(\Gamma_N) \subset L^2(\Gamma_N)$$

with dense, bounded and proper injections. (**Hint.** $1 \notin \widetilde{H}^{1/2}(\Gamma_N)$.)

The dual spaces for the two possible trace spaces on Γ_N are defined so that the following

$$H^{1/2}(\Gamma_N) \subset L^2(\Gamma_N) \subset \widetilde{H}^{-1/2}(\Gamma_N) \qquad \widetilde{H}^{1/2}(\Gamma_N) \subset L^2(\Gamma_N) \subset H^{-1/2}(\Gamma_N)$$

are Gelfand triples. We will formally write

$$\widetilde{H}^{-1/2}(\Gamma_N) := H^{1/2}(\Gamma_N)', \qquad H^{-1/2}(\Gamma_N) := \widetilde{H}^{1/2}(\Gamma_N)'.$$

(e) Show that the expression

$$\langle (\mathbf{p}\cdot\mathbf{n})|_{\Gamma_N}, \xi\rangle_{H^{-1/2}(\Gamma_N)\times\widetilde{H}^{1/2}(\Gamma_N)} := (\mathbf{p}, \nabla v)_\Omega + (\nabla\cdot\mathbf{p}, v)_\Omega \qquad v \in V_D, \quad \gamma v|_{\Gamma_N} = \xi$$

defines a bounded linear map $\mathbf{H}(\mathrm{div},\Omega) \to H^{-1/2}(\Gamma_N)$.

(f) Show that

$$\langle (\mathbf{p}\cdot\mathbf{n})|_{\Gamma_N}, \xi\rangle_{H^{-1/2}(\Gamma_N)\times\widetilde{H}^{1/2}(\Gamma_N)} = \langle \mathbf{p}\cdot\mathbf{n}, \widetilde{\xi}\rangle_\Gamma,$$

where $\widetilde{\xi}$ is the extension by zero of ξ.

(**Note about terminology.** In much of the literature, the space $\widetilde{H}^{1/2}(\Gamma_N)$ is denoted $H^{1/2}_{00}(\Gamma_N)$. Confusion reigns when denoting duals.)

7. **Mixed problems for the Laplacian.** Let us place ourselves in the notation of the previous problem. Let

$$f \in L^2(\Omega), \qquad h \in H^{-1/2}(\Gamma_N).$$

(a) Write the variational problem that is equivalent to the following minimization problem:

$$\tfrac{1}{2}\|\nabla u\|^2_\Omega - (f,u)_\Omega - \langle h, (\gamma u)|_{\Gamma_N}\rangle_{H^{-1/2}(\Gamma_N)\times\widetilde{H}^{1/2}(\Gamma_N)} = \min!, \qquad u \in V_D.$$

(b) Show well-posedness of the previous variational problem.

(c) Write an equivalent boundary value problem.

(d) Take now

$$f \in L^2(\Omega), \qquad g \in H^{1/2}(\Gamma_D), \qquad h \in H^{-1/2}(\Gamma_N).$$

Study the problem

$$\left[\begin{array}{l} u \in H^1(\Omega), \\ \gamma u|_{\Gamma_D} = g, \\ (\nabla u, \nabla v)_\Omega = (f,u)_\Omega + \langle h, (\gamma v)|_{\Gamma_N}\rangle_{H^{-1/2}(\Gamma_N)\times\widetilde{H}^{1/2}(\Gamma_N)} \qquad \forall v \in V_D. \end{array}\right.$$