MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Problems VI. Compact perturbations of coercive problems

1. Let Ω be a Lipschitz domain, and $\kappa \in L^\infty(\Omega)$ satisfy $\kappa \ge \kappa_0 > 0$ almost everywhere. Let finally $c \in L^\infty(\Omega)$ and $\alpha \in L^\infty(\Gamma)$. (No sign conditions are assumed on these two coefficients.) Show that the BVP

$$\left[\begin{array}{ll} -\nabla\cdot(\kappa\nabla u) + cu = f & \text{in } \Omega, \\ \kappa\nabla u\cdot\mathbf{n} + \alpha\gamma u = h & (\text{on } \Gamma), \end{array}\right.$$

is well-posed (with arbitrary data $f \in L^2(\Omega)$, $h \in H^{-1/2}(\Gamma)$) if and only if

$$\left.\begin{array}{rl} -\nabla\cdot(\kappa\nabla u) + cu = 0 & \text{in } \Omega \\ \kappa\nabla u\cdot\mathbf{n} + \alpha\gamma u = 0 & (\text{on } \Gamma) \end{array}\right] \Longrightarrow u = 0.$$

2. Let Ω be a Lipschitz domain and $\kappa \in L^\infty(\Omega)$ satisfy $\kappa \ge \kappa_0 > 0$ almost everywhere. Let $\boldsymbol\beta \in L^\infty(\Omega)^d$ and $c \in L^\infty(\Omega)$. Show that the problem

$$\left[\begin{array}{ll} -\nabla\cdot(\kappa\nabla u) + \boldsymbol\beta\cdot\nabla u + c\,u = f & \text{in } \Omega, \\ \gamma u = g, & \end{array}\right.$$

is well-posed (data are arbitrary functions $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$) if and only if

$$\left.\begin{array}{rl} -\nabla\cdot(\kappa\nabla u) + \boldsymbol\beta\cdot\nabla u + c\,u = f & \text{in } \Omega \\ \gamma u = g & \end{array}\right] \Longrightarrow u = 0.$$

3. **Finite dimensionality of eigenfunction spaces.** Let Ω be a Lipschitz domain. A Dirichlet eigenvalue of the Laplacian in Ω is $\lambda \in \mathbb{C}$ such that the problem

$$\left[\begin{array}{ll} -\Delta u = \lambda u & \text{in } \Omega, \\ \gamma u = 0, & \end{array}\right.$$

has non-trivial solutions. Show that the set of solutions of this problem is finite dimensional. Repeat the argument for Neumann eigenfunctions, that is, solutions of

$$\left[\begin{array}{ll} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_\nu u = 0. & \end{array}\right.$$

(**Hint.** Rewrite the problem in a form where you can apply the Fredholm alternative.)

4. Consider the unit ball $B(\mathbf{0};1) \subset \mathbb{R}^2$ and the sets $\Omega_j := B(\mathbf{0};1) \setminus \Xi_j$, where

$$\Xi_1 := (-\tfrac{1}{2}, \tfrac{1}{2}) \times \{0\}, \qquad \Xi_2 := (0,1) \times \{0\}.$$

Show that the Rellich-Kondrachov theorem holds in these sets, that is, that the space $H^1(\Omega_j)$ is compactly embedded into $L^2(\Omega_j)$. (**Hint.** Separate the unit ball into a positive and negative part and use the continuity of the restriction operators.)

5. Assume that $\Omega = \Omega_1 \cup \Omega_N$, where all the domains Ω_j are Lipschitz. Show that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$.

6. **A compactness result for the trace operator.** Let $Q := (0,1)^d$, $\square := (0,1)^{d-1} \equiv (0,1)^{d-1} \times \{0\}$, and $\gamma : H^1(Q) \to L^2(\square)$ be the associated trace operator. The goal of this exercise is the proof of the compactness of γ. Consider the functions

$$\phi_\alpha(\mathbf{x}) := \prod_{j=1}^{d} \cos(\alpha_j \pi x_j) \quad \alpha \in \mathbb{N}^d, \qquad \psi_\beta(\mathbf{x}) := \prod_{j=1}^{d-1} \cos(\beta_j \pi x_j) \qquad \beta \in \mathbb{N}^{d-1}.$$

Note that

$$\gamma \phi_{(\beta,m)} = \psi_\beta \qquad \beta \in \mathbb{N}^{d-1}, m \geq 0.$$

Consider finally the projection

$$P_\beta u := \sum_{m=0}^{\infty} \frac{(u, \phi_{(\beta,m)})_{H^1(Q)}}{\|\phi_{(\beta,m)}\|^2_{H^1(Q)}} \phi_{(\beta,m)}$$

(a) Show that

$$u = \sum_{\beta \in \mathbb{N}^{d-1}} P_\beta u \qquad \text{in } H^1(Q), \qquad \forall u \in H^1(Q).$$

(Note that the series is an orthogonal series.)

(b) Show that the operator $\gamma P_\beta : H^1(Q) \to L^2(\square)$ is compact.

(c) Show that

$$\|\gamma u - \sum_{\|\beta\| \leq N} \gamma P_\beta u\|^2_\square \leq C_N \|u\|^2_{H^1(Q)} \qquad \forall u \in H^1(Q),$$

where $C_N \to 0$ as $N \to \infty$. Prove that $\gamma : H^1(Q) \to L^2(\square)$ is compact.

7. Using local charts and the previous exercise prove that for all bounded Lipschitz domain Ω, the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is compact. (**Hint.** You will need to rescale the reuslt of the previous exercise to make the reference half-cylinder fit into a parallelepiped.)

8. Show that the inclusion operator $H^{1/2}(\Gamma) \to L^2(\Gamma)$ is compact. (**Hint.** Use a lifting of the trace.)

9. Let Ω be a bounded Lipschitz set with boundary Γ, and let $H^{3/2}(\Gamma) := \{\xi \in L^2(\Gamma) : \xi = \gamma u, \quad u \in H^2(\Omega)\}$. Show that $H^{3/2}(\Gamma)$ is compactly embedded into $H^{1/2}(\Gamma)$.