MATH 836: Elliptic Partial Differential Equations

Spring 2013 (F.–J. Sayas) Problems VII. Eigenvalues of elliptic operators

1. **One-dimensional eigenvalue problems.** Extract all the information that the Hilbert-Schmidt theory provides on the following eigenvalue problems. In particular, characterize Sobolev spaces in terms of the corresponding Fourier series. (Note that all eigenvalues and eigenfunctions can be computed in these cases and the theory gives additional insight on the convergence of the different Fourier series.)

   (a) One dimensional Dirichlet eigenvalues and sine series.

   $$-u'' = \lambda u \quad \text{in } (0,1), \qquad u(0) = u(1) = 0.$$

   (b) One dimensional Neumann eigenvalues and cosine series.

   $$-u'' = \lambda u \quad \text{in } (0,1), \qquad u'(0) = u'(1) = 0.$$

   (c) One dimensional mixed eigenvalues and half-sine series.

   $$-u'' = \lambda u \quad \text{in } (0,1), \qquad u(0) = u'(1) = 0.$$

   (d) Periodic problem and sine-and-cosine series.

   $$-u'' = \lambda u \qquad u(0) = u(1), \qquad u'(0) = u'(1).$$

2. **Series solution for the Helmholtz equation.** Let $\{\lambda_n; \phi_n\}$ be a complete orthonormal eigensystem for the Laplacian on a bounded domain with Dirichlet boundary conditions

   $$\phi_n \in H_0^1(\Omega), \qquad -\Delta\phi_n = \lambda_n \phi_n, \qquad (\phi_n, \phi_m)_\Omega + \delta_{nm}.$$

   Let $k^2 =\neq \lambda_n$ for all n. Show that for $f \in L^2(\Omega)$, the series

   $$u = \sum_{n=1}^\infty \frac{1}{k^2 - \lambda_n}(f, \phi_n)_\Omega \phi_n$$

   converges in $H_0^1(\Omega)$ to the solution of

   $$\Delta u + k^2 u = f, \qquad u \in H_0^1(\Omega).$$

3. **A reciprocal of the Hilbert-Schmidt theorem.** Let μ_n be a non-increasing sequence of positive real numbers converging to zero. Let $\{\phi_n\}$ be an X-orthonormal sequence. Show that the series

   $$\sum_{n=1}^\infty \mu_n (\cdot, \phi_n)_X \phi_n$$

   converges in the space of bounded linear operators $X \to X$ to a compact, selfadjoint and positive definite operator. Show that $N(G)$ is the orthogonal of $\text{span}\{\phi_n : n \geq 1\}$.

4. **The singular value decomposition.** Let X, Y be Hilbert spaces and $G : X \to Y$ be a compact operator such that $N(G)^\top$ is infinitely dimensional.

   (a) Show that $N(G^*G) = N(G)$ and $N(GG^*) = N(G^*)$.

   (b) Show that we can find an X-orthonormal sequence, $\{\phi_n\}$, and a sequence of positive non-increasing numbers, converging to zero, such that

   $$G^*G = \sum_{n=1}^{\infty} \mu_n (\cdot, \phi_n)_X \phi_n,$$

   with convergence in the sense of bounded operators $X \to X$.

   (c) Let now $\sigma_n := \sqrt{\mu_n}$ and $\psi_n := \sigma_n^{-1} G\phi_n$. Show that

   $$G^*\psi_n = \sigma_n \phi_n, \qquad G\phi_n = \sigma_n \psi_n, \qquad GG^*\psi_n = \mu_n \psi_n.$$

   Prove that $\{\psi_n\}$ is $Y-$orthonormal.

   (d) Let now

   $$R := G - \sum_{n=1}^{\infty} \sigma_n (\cdot, \phi_n)_X \psi_n.$$

   Whos that R is well defined and it is a compact operator $X \to Y$. Show that $G^*R = 0$ and that $R\phi \perp N(G^*)$ for all ϕ. From this, prove that $R = 0$, that is,

   $$G = \sum_{n=1}^{\infty} \sigma_n (\cdot, \phi_n)_X \psi_n.$$

   This decomposition is called the SVD of G.

   (e) Show that

   $$G^* = \sum_{n=1}^{\infty} \sigma_n (\cdot, \psi_n)_Y \phi_n.$$

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Spring 2013 (F.–J. Sayas)   Problems   VI. Compact perturbations of coercive problems

1. Let Ω be a Lipschitz domain, and $\kappa \in L^\infty(\Omega)$ satisfy $\kappa \ge \kappa_0 > 0$ almost everywhere. Let finally $c \in L^\infty(\Omega)$ and $\alpha \in L^\infty(\Gamma)$. (No sign conditions are assumed on these two coefficients.) Show that the BVP

$$\left[\begin{array}{ll} -\nabla\cdot(\kappa\nabla u) + cu = f & \text{in } \Omega, \\ \kappa\nabla u\cdot\mathbf{n} + \alpha\gamma u = h & (\text{on } \Gamma), \end{array}\right.$$

is well-posed (with arbitrary data $f \in L^2(\Omega)$, $h \in H^{-1/2}(\Gamma)$) if and only if

$$\left.\begin{array}{rl} -\nabla\cdot(\kappa\nabla u) + cu = 0 & \text{in } \Omega \\ \kappa\nabla u\cdot\mathbf{n} + \alpha\gamma u = 0 & (\text{on } \Gamma) \end{array}\right] \Longrightarrow u = 0.$$

2. Let Ω be a Lipschitz domain and $\kappa \in L^\infty(\Omega)$ satisfy $\kappa \ge \kappa_0 > 0$ almost everywhere. Let $\boldsymbol{\beta} \in L^\infty(\Omega)^d$ and $c \in L^\infty(\Omega)$. Show that the problem

$$\left[\begin{array}{ll} -\nabla\cdot(\kappa\nabla u) + \boldsymbol{\beta}\cdot\nabla u + c\,u = f & \text{in } \Omega, \\ \gamma u = g, & \end{array}\right.$$

is well-posed (data are arbitrary functions $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$) if and only if

$$\left.\begin{array}{rl} -\nabla\cdot(\kappa\nabla u) + \boldsymbol{\beta}\cdot\nabla u + c\,u = f & \text{in } \Omega \\ \gamma u = g & \end{array}\right] \Longrightarrow u = 0.$$

3. **Finite dimensionality of eigenfunction spaces.** Let Ω be a Lipschitz domain. A Dirichlet eigenvalue of the Laplacian in Ω is $\lambda \in \mathbb{C}$ such that the problem

$$\left[\begin{array}{ll} -\Delta u = \lambda u & \text{in } \Omega, \\ \gamma u = 0, & \end{array}\right.$$

has non-trivial solutions. Show that the set of solutions of this problem is finite dimensional. Repeat the argument for Neumann eigenfunctions, that is, solutions of

$$\left[\begin{array}{ll} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_\nu u = 0. & \end{array}\right.$$

(**Hint.** Rewrite the problem in a form where you can apply the Fredholm alternative.)

4. Consider the unit ball $B(\mathbf{0};1) \subset \mathbb{R}^2$ and the sets $\Omega_j := B(\mathbf{0};1)\setminus \Xi_j$, where

$$\Xi_1 := (-\tfrac{1}{2},\tfrac{1}{2})\times\{0\}, \qquad \Xi_2 := (0,1)\times\{0\}.$$

Show that the Rellich-Kondrachov theorem holds in these sets, that is, that the space $H^1(\Omega_j)$ is compactly embedded into $L^2(\Omega_j)$. (**Hint.** Separate the unit ball into a positive and negative part and use the continuity of the restriction operators.)

5. Assume that $\Omega = \Omega_1 \cup \ldots \cup \Omega_N$, where all the domains Ω_j are Lipschitz. Show that $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$.

6. **A compactness result for the trace operator.** Let $Q := (0,1)^d$, $\Box := (0,1)^{d-1} \equiv (0,1)^{d-1} \times \{0\}$, and $\gamma : H^1(Q) \to L^2(\Box)$ be the associated trace operator. The goal of this exercise is the proof of the compactness of γ. Consider the functions

$$\phi_\alpha(\mathbf{x}) := \prod_{j=1}^{d} \cos(\alpha_j \pi x_j) \quad \alpha \in \mathbb{N}^d, \qquad \psi_\beta(\mathbf{x}) := \prod_{j=1}^{d-1} \cos(\beta_j \pi x_j) \qquad \beta \in \mathbb{N}^{d-1}.$$

Note that

$$\gamma \phi_{(\beta,m)} = \psi_\beta \qquad \beta \in \mathbb{N}^{d-1}, m \geq 0.$$

Consider finally the projection

$$P_\beta u := \sum_{m=0}^{\infty} \frac{(u, \phi_{(\beta,m)})_{H^1(Q)}}{\|\phi_{(\beta,m)}\|^2_{H^1(Q)}} \phi_{(\beta,m)}$$

(a) Show that

$$u = \sum_{\beta \in \mathbb{N}^{d-1}} P_\beta u \qquad \text{in } H^1(Q), \qquad \forall u \in H^1(Q).$$

(Note that the series is an orthogonal series.)

(b) Show that the operator $\gamma P_\beta : H^1(Q) \to L^2(\Box)$ is compact.

(c) Show that

$$\|\gamma u - \sum_{\|\beta\| \leq N} \gamma P_\beta u\|^2_\Box \leq C_N \|u\|^2_{H^1(Q)} \qquad \forall u \in H^1(Q),$$

where $C_N \to 0$ as $N \to \infty$. Prove that $\gamma : H^1(Q) \to L^2(\Box)$ is compact.

7. Using local charts and the previous exercise prove that for all bounded Lipschitz domain Ω, the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is compact. (**Hint.** You will need to rescale the result of the previous exercise to make the reference half-cylinder fit into a parallelepiped.)

8. Show that the inclusion operator $H^{1/2}(\Gamma) \to L^2(\Gamma)$ is compact. (**Hint.** Use a lifting of the trace.)

9. Let Ω be a bounded Lipschitz set with boundary Γ, and let $H^{3/2}(\Gamma) := \{\xi \in L^2(\Gamma) : \xi = \gamma u, \quad u \in H^2(\Omega)\}$, endowed with the image norm. Show that $H^{3/2}(\Gamma)$ is compactly embedded into $H^{1/2}(\Gamma)$.